Lectures on the Theory of Reciprocants.

BY PROFESSOR SYLVESTER, F. R. S., *Savilian Professor of Geometry in the University of Oxford.*

[Reported by JAMES HAMMOND, M. A.]

LECTURE XVII.

The fundamental reciprocants for extent 3, given in the last lecture, agree with the irreducible invariants of a binary cubic both in number and type, with the single exception that the degree of the cubic discriminant is lower by unity than that of the reciprocant corresponding to it. When the extent is raised to 4, both the discriminant and its analogue cease to rank among the irreducible forms, the former being expressible as a rational integral function of invariants of lower degree, and the latter as a similar function of reciprocants. But the increase of extent introduces three additional reciprocants whose leading terms are a^2e, a^2ce and a^3a^2, whereas the additional invariants are only two in number and begin with ae and ace respectively.

The irreducible reciprocants of extent 4 are as follows:

deg. wt.

1.0 $\quad a,$

2.2 $\quad 4M \;= 4ac - 5b^2,$

3.3 $\quad A \;= a^2d - 3abc + 2b^3,$

3.4 $\quad P_4 \;= 50a^2e - 175abd + 28ac^2 + 105b^2c,$*

4.6 $\quad (a^2ce) = 800a^2ce - 1000ab^2e - 875a^2d^2 + 2450abcd - 1344ac^3 - 35b^2c^2,$

5.8 $\quad (a^3e^2) = 625a^3e^2 - 4375a^2bde - 49700a^2c^2e + 128625ab^2ce - 78750b^4e$
$\quad + 55125a^2cd^2 - 61250ab^2d^2 - 156800abc^2d + 183750b^3cd$
$\quad + 84868ac^4 - 102165b^2c^3.$

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* P_4 is the protomorph of minimum degree; the other protomorph, B, which will be used when we treat of Principiants, is, when expressed in terms of the irreducible forms,

$$B = \frac{1}{50}(aP_4 - 128M^2).$$

The similar list of invariants for the quartic is

deg. wt.	
1.0	a,
2.2	$ac - b^2$,
3.3	$a^2d - 3abc + 2b^3$,
2.4	$ae - 4bd + 3c^2$,
3.6	$ace - b^2e - ad^2 + 2bcd - c^3$.

To obtain the fundamental forms of extent 4 we have to combine M, A and the Quasi-Discriminant

$$(a^3d^2) = 125a^3d^2 - 750a^2bcd + 500ab^3d + 256a^2c^3 + 165ab^2c^2 - 300b^4c$$

with the additional Protomorph

$$P_4 = 50a^2e - 175abd + 28ac^2 + 105b^2c$$

in such a manner that the combination contains a factor a. The removal of this factor gives rise to a form of lower degree, and the process is repeated as often as possible.

Calling that portion of any form which does not contain a its residue, the residue of $4M$ is $-5b^2$, that of (a^3d^2) being $-300b^4c$, and that of P_4 being $105b^2c$. Thus $$16MP_4 - 7(a^3d^2)$$ contains the factor a, and leads to (a^2ce) of the type 6; 4, 4, which is the analogue to the Catalecticant

$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}.$$

The form (a^3d^2) now ceases to be a groundform (= irreducible form) and is replaced by the Quasi-Catalecticant (a^2ce), for

$$(a^3d^2) = \frac{16}{7} MP_4 - \frac{1}{7} a(a^2ce).$$

Similarly, the Cubic Discriminant, a groundform *quâ* the letters a, b, c, d, becomes reducible when a new letter, e, is introduced, and is then replaced by the Catalecticant.

We now come to an extra form which has no analogue in invariants. The residue of the Quasi-Catalecticant (a^2ce) is $-35b^2c^2$, and consequently

$$P_4^2 - 252M(a^2ce)$$

divides by a numerical multiple of a (as it happens by $4a$) and yields the form (a^3e^2), whose type is 8; 5, 4.

Here the deduction of new fundamental forms comes to an end on account of the appearance of e in the residue of (a^3e^2). It would have ended sooner but for the apparently accidental non-appearance of the term b^3d (of the same type 6; 4, 4 as b^2c^2) in the residue of (a^2ce). Had this term appeared, no combination could have been made leading to a new groundform after (a^2ce). We are able to show from *à priori* considerations that it cannot exist.

For the arguments in the annihilator V, up to ∂_e inclusive, are

$$a^2\partial_b,\ ab\partial_c,\ ac\partial_d,\ b^2\partial_d,\ ad\partial_e,\ \text{and}\ bc\partial_e.$$

If, now, the term μb^3d were to form part of a Pure Reciprocant, $b^2\partial_d$ operating upon it would give μb^5; but every other portion of the operator would necessarily give terms containing one or other of the letters a, c. Since such terms cannot destroy μb^5, we must have $\mu b^5 = 0$. Hence the term in question is necessarily non-existent.

The method of combining the protomorphs which we have followed shows that the fundamental reciprocants of extent 4 are connected *inter se* by the two relations or syzygies

$$7\,(256M^3 + 125A^2) - 16aMP_4 + a^2\,(a^2ce) = 0,$$
$$P_4^2 - 252M(a^2ce) - 4a\,(a^3e^2) = 0.$$

The invariants of the binary quartic are connected by only one syzygy, similar to the first of these; the second has no analogue in the theory of Invariants. It has been shown that the irreducible reciprocants of extent 3 are connected by the syzygy

$$256M^3 + 125A^2 - a\,(a^3d^2) = 0.$$

Substituting in this for the Quasi-Discriminant (a^3d^2) its value expressed in terms of the fundamental forms of extent 4, by means of the equation

$$16MP_4 - 7\,(a^3d^2) = a\,(a^2ce),$$

we obtain the first of the above syzygies. By a precisely similar substitution, the syzygy connecting the invariants of the quartic is derived from the one which connects the invariants of the cubic.

Every reciprocant of extent 4 is a rational integral function of the six fundamental forms given in the table; and, by means of the syzygies, powers, but not products, of A and P_4 can be removed from this function. For the first syzygy gives A^2 and the second gives P_4^2 as a rational integral function of the

four remaining forms a, M, (a^2ce), and (a^3e^2). Hence every reciprocant of extent 4 is of one or other of the forms

$$\Phi,\ A\Phi,\ P_4\Phi,\ AP_4\Phi,$$

where Φ does not contain either A or P_4, but is a rational integral function of the other four fundamental forms.

Let the four forms which appear in Φ occur raised to the powers κ, λ, μ, ν, respectively, in one of its terms. Since the degree-weights of these four forms are 1.0, 2.2, 4.6 and 5.8,

any such term may be represented by

$$a^\kappa (a^2x^2)^\lambda (a^4x^6)^\mu (a^5x^8)^\nu.$$

Thus the totality of the terms in Φ will be represented by

$$\Sigma a^\kappa (a^2x^2)^\lambda (a^4x^6)^\mu (a^5x^8)^\nu = \frac{1}{(1-a)(1-a^2x^2)(1-a^4x^6)(1-a^5x^8)}.$$

Now, A, P_4 and AP_4 have the degree-weights

$$3.3,\ 3.4 \text{ and } 6.7,$$

and consequently the totality of terms in

$$\Phi,\ A\Phi,\ P_4\Phi \text{ and } AP_4\Phi$$

(*i. e.* the totality of the pure reciprocants of extent 4) will be represented by

$$(1+a^3x^3+a^3x^4+a^6x^7)\,\Sigma a^\kappa (a^2x^2)^\lambda (a^4x^6)^\mu (a^5x^8)^\nu$$
$$= \frac{1+a^3x^3+a^3x^4+a^6x^7}{(1-a)(1-a^2x^2)(1-a^4x^6)(1-a^5x^8)}.$$

Hence the number of Pure Reciprocants of the type w; i, 4 is the coefficient of a^ix^w in the expansion of a fraction whose numerator is

$$1+a^3x^3+a^3x^4+a^6x^7,$$

with the denominator

$$(1-a)(1-a^2x^2)(1-a^4x^6)(1-a^5x^8).$$

This fraction is called the Representative Form of the Generating Function, in contradistinction to the Crude Form, which is a fraction with the numerator

$$1-a^{-1}x,$$

having for its denominator

$$(1-a)(1-ax)(1-ax^2)(1-ax^3)(1-ax^4).$$

The crude form expresses the fact that the number of pure reciprocants of the

type $$w;\ i,j$$
is $$(w;\ i,j)-(w-1;\ i+1,j).$$
Its numerator is $1-a^{-1}x$ for all extents; for the general case in which the extent is j, its denominator consists of the $j+1$ factors
$$(1-a)(1-ax)(1-ax^2)\ldots\ldots(1-ax^j).$$

The removal of the negative terms [corresponding to cases in which $(w;i,j)<(w-1;\ i+1,j)$] from the crude form would give either the representative form or one equivalent to it, according as the representative form is or is not in its lowest terms. In the parallel theory of Invariants the terms to be rejected are those for which $ij-2w<0$; but we do not at present know of any similar criterion for reciprocants, and are thus unable to pass directly from the crude to the representative form of their generating function.

Knowing both the crude and the representative form for reciprocants of extent 4, we may verify that the difference between these two forms of the generating function is omninegative. It will be found that
$$\frac{1-a^{-1}x}{(1-a)(1-ax)(1-ax^2)(1-ax^3)(1-ax^4)}$$
$$=\frac{1+a^3x^3+a^3x^4+a^6x^7}{(1-a)(1-a^2x^2)(1-a^4x^6)(1-a^5x^8)}$$
$$-\frac{1}{(1-ax^2)(1-ax^3)(1-ax^4)}\left(\frac{a^{-1}x+a^2x^5}{1-a^4x^6}+\frac{x^2+a^2x^6}{1-a^5x^8}\right)$$
$$-\frac{1}{(1-ax^4)(1-a^4x^6)(1-a^5x^8)}\left(\frac{x+a^5x^{10}}{1-ax^2}+\frac{a^3x^5+a^2x^7}{1-ax^3}\right).$$
Thus the crude form is seen to consist of an omnipositive part, equal to the representative form, and an omninegative part.

There is no difficulty in obtaining the representative form of the generating function for pure reciprocants of extents 2 and 3. In the one case every reciprocant is a rational integral function of two forms of degree-weight, 1.0 and 2.2 respectively. The generating function is therefore
$$\frac{1}{(1-a)(1-a^2x^2)}.$$
In the other case (*i. e.* for extent 3) every pure reciprocant can be expressed as a rational integral function of four forms, of which the degree-weights are 1.0, 2.2, 3.3 and 5.6, no higher power than the first of the form 3.3 occurring in the function. Thus the representative form is
$$\frac{1+a^3x^3}{(1-a)(1-a^2x^2)(1-a^5x^6)}.$$

LECTURE XVIII.

The number of Pure Reciprocants of a given degree is finite; the number of Invariants of the same degree is infinite. Thus, for example, we have the well-known series of invariants

$$ac - b^2,\ ae - 4bd + 3c^2, \ldots\ldots,$$

all of degree 2, but of weights and extents proceeding to infinity. This may be proved from the theory of partitions (see *American Journal of Mathematics*, Vol. V, No. 1, On Subinvariants, Excursus on Rational Fractions and Partitions). It will be seen in that article that if $N(w:i)$ is the number of ways in which w can be divided into i parts, and if P is the least common multiple of $2, 3, 4, \ldots\ldots i$, then $N(w:i)$ can be expressed under the form

$$F(w,\ i) + F'(w,\ i,\ p),$$

where p is the residue of w in respect of P.

Writing
$$w + \frac{i(i+1)}{4} = \nu,$$

$F(w,\ i)$ is of the form
$$\frac{\nu^{i-1}}{2^2.3^2.\ldots\ldots(i-1)^2.i} + \ldots\ldots,$$

all the succeeding indices of the powers of ν in $F(w,\ i)$ decreasing by 2, and their coefficients being transcendental functions of i which involve Bernoulli's Numbers.

In $F'(w,\ i,\ p)$ the highest index of ν is one unit less than the number of times that i is divisible by 2, *i. e.* is $\frac{i-2}{2}$ or $\frac{i-3}{2}$, according as i is even or odd.

Thus, for the partitions of w into 3 parts, we have the formula

$$N(w:3) = \left\{\frac{\nu^2}{12} - \frac{7}{72}\right\} + \left\{\frac{1}{8}(-1)^{\nu+1} + \frac{1}{9}(\rho_1^\nu + \rho_2^\nu)\right\},$$

where $\nu = w + \frac{1+2+3}{2} = w + 3$.

And, for the partitions of w into 4 parts,

$$N(w:4) = \left\{\frac{\nu^3}{144} - \frac{5\nu}{96}\right\} + \left\{\frac{1}{32}(-)^{\nu+1}\nu + \frac{1}{27}(\rho_1^{\nu+1} + \rho_2^{\nu+1} - \rho_1^{\nu-1} - \rho_2^{\nu-1}) - \frac{1}{32}(i_1^{\nu+1} + i_2^{\nu+1} - i_1^{\nu-1} - i_2^{\nu-1})\right\},$$

where $\nu = w + \frac{1+2+3+4}{2} = w + 5$,

and ρ_1, ρ_2 are the roots of $\rho^2+\rho+1=0$,
i_1, i_2 " " " " $i^2+1=0$;

in other words, ρ_1 and ρ_2 are primitive cube roots, and i_1, i_2 primitive fourth roots of unity.

The principal term of $N(w:3)$, regarded as a function of w, is

$$\frac{w^2}{12}=\frac{w^2}{2^2.3}, \text{ that of } N(w:4) \text{ being } \frac{w^3}{144}=\frac{w^3}{2^2.3^2.4}.$$

And in general the principal term of $N(w:i)$ is

$$\frac{w^{i-1}}{2^2.3^2.4^2.\dots.(i-1)^2.i}.$$

Hence it follows, from a general algebraical principle, that for all values of w above a certain limit, which depends on the value of i and may be determined by the aid of partition tables, $(w;i,\infty)-(w-1;i+1,\infty)$ must become negative.

Ultimately, $\dfrac{(w-1;i+1,\infty)}{(w;i,\infty)}=\dfrac{w}{i(i+1)}$, which must eventually be greater than unity. This shows that beyond a certain value of w there can be no pure reciprocant, and consequently that the number of pure reciprocants of a given degree i is finite.

Mr. Hammond remarks that the formulae for $N(w:3)$ and $N(w:4)$ may, by the substitution of trigonometrical expressions for the roots of unity, accompanied by some easy reductions, be transformed into

$$N(w:3)=\frac{\nu^2}{12}+\frac{1}{4}\sin^2\frac{\nu\pi}{2}-\frac{4}{9}\sin^2\frac{\nu\pi}{3}$$

and
$$N(w:4)=\frac{\nu^3}{144}-\frac{\nu}{12}+\frac{\nu}{16}\sin^2\frac{\nu\pi}{2}+\frac{1}{8}\sin\frac{\nu\pi}{2}-\frac{2}{9\sqrt{3}}\sin\frac{\nu\pi}{3},$$

where, in the first formula, $\nu=w+3$, and in the second $\nu=w+5$. He also obtains the principal term of $N(w:i)$ from first principles as follows:

The partitions of w into i parts may be separated into two sets, the first containing at least one zero part in each of its partitions, the second consisting of partitions in which no zero part occurs.

Suppressing one zero part in each partition of the first set, we see that the number of partitions in which 0 occurs is $N(w:i-1)$. Diminishing each part by unity in those partitions which contain no zeros, their number is seen to be $N(w-i:i)$. The sum of these two numbers is $N(w:i)$, which is the total

number of partitions, and consequently $N(w:i) = N(w:i-1) + N(w-i:i)$. Let the principal term of $N(w:i-1)$ be aw^{i-2}, where a is independent of w, and write $w = ix$, $N(w:i) = u_x$, $N(w-i:i) = u_{x-1}$.

Then $u_x - u_{x-1} = aw^{i-2} + \ldots\ldots = ai^{i-2}x^{i-2} + \ldots\ldots$

Hence, by a simple summation, we find

$$u_x = ai^{i-2}\{x^{i-2} + (x-1)^{i-2} + (x-2)^{i-2} + \ldots\ldots\} + \ldots\ldots$$

But, since only the principal term of u_x is required, this summation may be replaced by an integration. Thus the principal term of u_x is

$$ai^{i-2}\int x^{i-2}dx = \frac{ai^{i-2}x^{i-1}}{i-1}.$$

Restoring $w = ix$ and $N(w:i) = u_x$,

we see that the principal term of $N(w:i)$ is $\frac{aw^{i-1}}{(i-1)i}$. Thus the principal term of $N(w:i)$ is found from that of $N(w:i-1)$ by multiplying it by $\frac{w}{(i-1)i}$.

When $i = 3$, the principal term is $\frac{w^2}{2^2.3}$; it is therefore $\frac{w^3}{2^2.3^2.4}$ when $i = 4$; and for the general case it is $\frac{w^{i-1}}{2^2.3^2.4^2.\ldots(i-1)^2.i}$.

The value of $N(w:i)$ is given in line i and column w of the following table:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	1	2	2	3	3	4	4	5	5	6	6	7	7	8
3	1	2	3	4	5	7	8	10	12	14	16	19	21	24
4	1	2	3	5	6	9	11	15	18	23	27	34	39	47
5	1	2	3	5	7	10	13	18	23	30	37	47	57	70
6	1	2	3	5	7	11	14	20	26	35	44	58	71	90

From an inspection of the tabulated values of $N(w:i)$ we see that

$N(w:2) - N(w-1:3)$ is negative or zero when $w > 2$,
$N(w:3) - N(w-1:4)$ " " " " " $w > 6$,
$N(w:4) - N(w-1:5)$ " " " " " $w > 8$,
$N(w:5) - N(w-1:6)$ " " " " " $w > 12$.

Hence for pure reciprocants of indefinite extent, whose degrees are

2, 3, 4, 5,

the highest possible weights are 2, 6, 8 and 12, respectively.

In like manner, from Euler's table, in his memoir De Partitione Numerorum (published in 1750), it will be found that

for degrees	2	3	4	5	6	7	8	9	10	11	12	13
the highest weights are	2	6	8	12	16	21	26	30	36	42	49	55

Further than this the table, which goes up to $w = 59$, will not enable us to proceed.

The actual number of pure reciprocants of degree i, weight w, and of indefinite extent, is seen in the following table, which gives the value of $N(w:i) - N(w-1:i+1)$ when positive, blank spaces being left in the table when this difference is zero or negative.

DEGREE $i =$ \ WEIGHT $w =$	2	3	4	5	6	7	8	9	10	11	12	13	14
2	1												
3	1	1	1		1								
4	1	1	2	1	2	1	2						
5	1	1	2	2	3	2	4	3	4	2	3		

Thus, for degree 2, there is only one pure reciprocant, viz.

$$(ac) = 4ac - 5b^2.$$

For degree 3 the table shows that, in addition to the compound form

$$a(ac) = a(4ac - 5b^2),$$

there are three others whose weights are 3, 4 and 6 respectively.

These are the three protomorphs,

$$(a^2d) = a^2d - 3abc + 2b^3,$$
$$(a^2e) = 50a^2e - 175abd + 28ac^2 + 105b^2c,$$
$$(a^2g) = 14a^2g - 63abf - 1350ace + 1782b^2e + 1470ad^2 - 4158bcd + 2310c^3.$$

With the above forms and a we are able to form the following compounds of degree 4: $a^2(ac)$, $a(a^2d)$, $(ac)^2$, $a(a^2e)$, $a(a^2g)$,
whose weights are 2, 3, 4, 4, 6.

The forms of degree 4 and weights 5, 7, 8, and one of the forms of weight 6, cannot be similarly made up of forms of inferior degree, and are therefore groundforms. Three of them are the protomorphs (a^3f), (a^3h) and (a^3i) of weights 5, 7 and 8, whose values were given in Lecture XVI. The groundform of weight 6 is the Quasi-Catalecticant given in the last lecture. All the

forms of degree 4 have thus been accounted for except one of the two forms of weight 8, which will be seen to be of extent 6, and to have a^2cg for its leading term.

We know from Euler's table that $N(8:4) - N(7:5) = 2$; *i. e.*

$$(8;4,8) - (7;5,8) = 2.$$

Now, $(8;4,7) = N(8:4) - 1$, the omitted partition being $8.0.0.0$,

$(8;4,6) = N(8:4) - 2$, the partition $7.1.0.0$ being also left out,

$(8;4,5) = N(8:4) - 4$, $\left\{\begin{array}{l}\text{for } 6.2.0.0 \text{ and } 6.1.1.0 \text{ are excluded from} \\ (8;4,5), \text{ but make their appearance in } (8;4,6).\end{array}\right.$

Similarly,

$$\begin{aligned}(7;5,7) &= N(7:5), \\ (7;5,6) &= N(7:5) - 1, \\ (7;5,5) &= N(7:5) - 2.\end{aligned}$$

We have, therefore,

$$\begin{aligned}(8;4,8) - (7;5,8) &= 2, \\ (8;4,7) - (7;5,7) &= 1, \\ (8;4,6) - (7;5,6) &= 1, \\ (8;4,5) - (7;5,5) &= 0.\end{aligned}$$

Hence we may draw the following inferences:

(1). No pure reciprocant exists whose type is $8;4,5$.

(2). The one whose type is $8;4,6$ must contain the letter g.

(3). No fresh form is found by making the extent 7 instead of 6, so that there is no pure reciprocant of weight 8 and degree 4 whose *actual extent* is 7.

(4). There is a pure reciprocant (the Protomorph whose leading term is a^3i) whose actual extent is 8.

(5). This, with the one whose actual extent is 6, makes up the two given by $(8;4,8) - (7;5,8) = 2$.

LECTURE XIX.

The following is a complete list of the irreducible reciprocants of indefinite extent for the degrees 2, 3 and 4:

Deg. wt.		
2.2	(ac),	
3.3	(a^2d),	
3.4	(a^2e),	
3.6	(a^2g),	
4.5	(a^3f),	
4.6	(a^2ce),	
4.7	(a^3h),	
4.8	(a^3i),	(a^2cg).

The values of all of them except (a^2cg) have been given in previous lectures, and the method of obtaining them sufficiently indicated. Thus (ac), (a^2d), (a^2e), (a^3f), (a^2g), (a^3h) and (a^3i) are the Protomorphs of minimum degree P_2, P_3, P_4, P_5, P_6, P_7 and P_8, respectively; and (a^2ce) is the Quasi-Catalecticant whose value has been set forth in the table of irreducible forms of extent 4. It will be remembered that (a^2ce) was found by combining the Quasi-Discriminant (a^3d^2) with P_2P_4 linearly in such a manner that the combination, which is of the 5th degree, divides by a and gives (a^2ce) of the 4th degree. If we try to find (a^2cg) by a similar process, it will be necessary to rise as high as the 7th degree, and then to drop down by successive divisions by a to the fourth.

In fact, since to a numerical factor près the residues of

	$P_2,\ P_3,\ P_4,\ P_5$
are	$b^2\ ,\ b^3\ ,\ b^2c,\ b^2c,$
that of	P_3P_5 will be b^6c,
and that of	$P_2^2P_4$ will be b^6c.

Thus a linear combination of P_3P_5 and $P_2^2P_4$ will be divisible by a, and, taking account of the numerical coefficients, we shall find

$$26P_2^2P_4 + 875P_3P_5 \equiv 0 \pmod{a}.$$

As a result of calculation, it will be seen that the above combination of the protomorphs divided by a,

$$\frac{1}{a}\left(26P_2^2P_4 + 875P_3P_5\right),$$

has (to a numerical factor près) the same residue as P_4^2.

Making a second combination and division by a, we find

$$7\left(\frac{26P_2^2P_4 + 875P_3P_5}{a}\right) - 25P_4^2 \equiv 0 \pmod{a} = aS, \text{ suppose.}$$

Then, by actual calculation, the residue of S is found to be

$$-262500b^4e + 612500b^3cd - 339080b^2c^3.$$

Two reductions have already been made in obtaining this form S of the 5th degree. A final combination of S with P_2P_6 and the form (a^3e^2), whose value was given in a former lecture, enables us to divide out once more by a and thus get the form (a^2cg) of the 4th degree.

It is the fact that P_2P_6 and (a^3e^2) have residues which are not the same to a numerical factor près which necessitates the long calculation above described.

No linear combination of P_2P_6 and (a^3e^2) with one another is divisible by a, and it is necessary to find a third form S a linear combination of which with both P_2P_6 and (a^3e^2) will divide by a.

There is, however, another way of arriving at the form (a^2cg) by using the eductive generator

$$G = 4(ac - b^2)\,\partial_b + 5(ad - bc)\,\partial_c + 6(ae - bd)\,\partial_d + \ldots.$$

Starting with the Quasi-Catalecticant

$$(a^2ce) = 800a^2ce - 1000ab^2e - 875a^2d^2 + 2450abcd - 1344ac^3 - 35b^2c^2,$$

and operating on it with G, we have

$$\begin{aligned} G(a^2ce) = {} & 4(ac - b^2)(-2000abe + 2450acd - 70bc^2) \\ & + 5(ad - bc)(800a^2e + 2450abd - 4032ac^2 - 70b^2c) \\ & + 6(ae - bd)(-1750a^2d + 2450abc) \\ & + 7(af - be)(800a^2c - 1000ab^2). \end{aligned}$$

The terms of this expression contain the common numerical factor 10, which may be rejected; thus we have

$$G(a^2ce) = 10(a^3cf),$$

where

$$\begin{aligned} (a^3cf) = {} & 560a^3cf - 700a^2b^2f - 650a^3de - 290a^2bce + 1500ab^3e \\ & + 2275a^2bd^2 - 1036a^2c^2d - 3710ab^2cd + 1988abc^3 + 63b^3c^2. \end{aligned}$$

This form (a^3cf) is the first educt of (a^2ce), and is irreducible (but, being of the fifth degree, does not appear in our list, which contains no forms of higher degree than the fourth). Operating on it with G, we obtain the educt of (a^3cf), which is the second educt of (a^2ce). This second educt will be of the 6th degree (its leading term will be a^4cg), but is reducible to the 5th when combined with

$$(4ac - 5b^2)(a^2ce),$$

as we know from the general theorem concerning the reduction of second educts. We shall thus obtain a form (a^3cg), the reduced second educt of (a^2ce), of the 5th degree, and a final combination of (a^3cg) with one or both of the forms P_2P_6 and (a^3e^2) will enable us to divide once more by a and thus arrive at (a^2cg) of the 4th degree.

By either of these methods we obtain

$$\begin{aligned} (a^2cg) = {} & 1176a^2cg - 8085a^2df + 7040a^2e^2 - 1470ab^2g + 18963abcf \\ & - 16940abde - 27160ac^2e + 26460acd^2 - 9555b^3f \\ & + 28098b^2ce + 12740b^2d^2 - 52822bc^2d + 21560c^4; \end{aligned}$$

but the second way, besides being more direct, gives us at the same time the value of the irreducible form (a^3cf).

Every Pure Reciprocant is an Invariant of a Binary Quantic whose coefficients $A, B, C, D, \ldots.$ are functions of the original elements $a, b, c, d, \ldots.$ such that

$$\begin{aligned} VA &= 0, \\ VB &= A, \\ VC &= 2B, \\ VD &= 3C, \\ &\ldots\ldots \end{aligned}$$

and conversely, every Invariant of this Binary Quantic, or of a system of such Binary Quantics, is a Pure Reciprocant.

This is a particular case of the more general theorem, due to Mr. Hammond, that if Θ is the operator,

$$\phi_1(a)\,\partial_b + \phi_2(a, b)\,\partial_c + \phi_3(a, b, c)\,\partial_d + \ldots.,$$

where $\phi_1, \phi_2, \phi_3, \ldots.$ are arbitrary rational integral functions, and if

$$A, B, C, D, \ldots., A', B', C', D', \ldots., A'', B'', C'', \ldots.$$

be any rational integral functions of the original letters $a, b, c, \ldots.$ which satisfy the conditions

$$\begin{array}{lll} \Theta A = 0, & \Theta A' = 0, & \Theta A'' = 0, \\ \Theta B = A, & \Theta B' = A', & \Theta B'' = A'', \\ \Theta C = 2B, & \Theta C' = 2B', & \Theta C'' = 2B'', \\ \Theta D = 3C, & \Theta D' = 3C', & \Theta D'' = 3C'', \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{array}$$

then every invariant in respect to the elements

$$A, B, C, D, \ldots., A', B', C', D', \ldots., A'', B'', C'', D'', \ldots.$$

is a rational integral solution of the equation

$$\Theta = 0.$$

Obviously, every rational integral solution of $\Theta = 0$ is an invariant in the above elements, so that the converse of the proposition is true. For the only conditions imposed upon $A, A', A'', \ldots.$ are that they shall be rational integral functions of $a, b, c, d, \ldots.$ annihilated by Θ. Let

$$\Phi(A, B, C, D, \ldots., A', B', C', D', \ldots., A'', B'', C'', D'', \ldots.)$$

be any invariant in the large letters. We have to show that

$$\Theta\Phi = 0.$$

Now,

$$\Theta\Phi = \frac{d\Phi}{dA}\Theta A + \frac{d\Phi}{dB}\Theta B + \frac{d\Phi}{dC}\Theta C + \ldots.$$
$$+ \frac{d\Phi}{dA'}\Theta A' + \frac{d\Phi}{dB'}\Theta B' + \frac{d\Phi}{dC'}\Theta C' + \ldots.$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence, writing for ΘA, ΘB, ΘC, ..., their values given above, we have

$$\Theta\Phi = (A\partial_B + 2B\partial_C + 3C\partial_D + \ldots.)\Phi$$
$$+ (A'\partial_{B'} + 2B'\partial_{C'} + 3C'\partial_{D'} + \ldots.)\Phi$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$= 0 \text{ (since } \Phi \text{ is an invariant)};$$

which proves the proposition.

Confining our attention to a single set of letters, the Binary Quantic

$$(A, B, C, \ldots. J, K, L \between x, y)^n,$$

whose coefficients are formed from one another by the successive operation of Θ as above, may be called a Quasi-Covariant; and it will follow immediately from the Theory of Binary Forms that every Covariant of a Quasi-Covariant is itself a Quasi-Covariant, and that every Invariant of any Quasi-Covariant (or system of Quasi-Covariants) is an Invariant in respect to the letters A, B, C, ..., and therefore, by what precedes, a rational integral solution of $\Theta = 0$.

Writing the terms of

$$(A, B, C, \ldots. J, K, L \between x, y)^n$$

in reverse order, we have

$$Ly^n + nKxy^{n-1} + \frac{n(n-1)}{1.2}Jxy^{n-2} + \ldots. + Ax^n,$$

where

$$\Theta L = nK,\ \Theta K = (n-1)J, \ldots. \Theta A = 0.$$

Thus the Quasi-Covariant may be written

$$Ly^n + \Theta Lxy^{n-1} + \frac{\Theta^2 L}{1.2}x^2y^{n-2} + \ldots. + \frac{\Theta^n L}{1.2.3\ldots.n}x^n = y^n\left(e^{\frac{x\Theta}{y}}\right)L,$$

where $\Theta^{n+1}L = 0$.

This is the general symbolic expression for a Quasi-Covariant. An example of a Quasi-Covariant has already been given in Lecture II (Vol. VIII, p. 205), where it was stated, and afterwards proved (p. 256), that the reciprocal of the n^{th} modified derivative could be put under the form

$$-t^{-n-3}\left(e^{-\frac{V}{t}}\right)a_n.$$

The numerator of this reciprocal expression, which may be called the reciprocal function, is

$$t^n \left(e^{-\frac{V}{t}}\right) a_n,$$

which is identical with the general expression

$$y^n \left(e^{\frac{x\Theta}{y}}\right) L,$$

if $$x = -1, \ y = t, \ L = a_n \text{ and } \Theta = V.$$

Hence every Invariant of the reciprocal function is a Pure Reciprocant.

This property of the reciprocal function was discovered independently by Mr. C. Leudesdorf, who published his results in the Proceedings of the London Mathematical Society (Vol. XVII, p. 208). Mr. Hammond's results were given in two letters to me dated January 15th and January 20th, 1886, and were briefly alluded to by him at a meeting of the London Mathematical Society. They are here published for the first time.

Recalling the form of the operator

$$\Theta = \phi_1(a)\,\partial_b + \phi_2(a, b)\,\partial_c + \phi_3(a, b, c)\,\partial_d + \ldots ,$$

where $\phi_1, \phi_2, \phi_3, \ldots$ are rational integral functions, we can form a Quasi-Covariant of extent j by a finite number of successive operations on a single letter of that extent.

To fix the ideas, take the letter d of extent 3, and operate on it with Θ; then

$$\Theta d = \phi_3(a, b, c).$$

Since $\phi_1, \phi_2, \phi_3, \ldots$ are by definition rational integral functions, we can, by operating a finite number of times with Θ, remove first c and then b from $\phi_3(a, b, c)$, and thus obtain

$$\Theta^n d = \text{funct. } a,$$

where n denotes a finite number of operations. Since $\Theta a = 0$, we have

$$\Theta^{n+1} d = 0.$$

In this manner we form the Quasi-Covariant of the n^{th} order

$$y^n \left(e^{\frac{x\Theta}{y}}\right) d.$$

If $\phi_2, \phi_3, \phi_4, \ldots$ do not contain higher powers than the first of the last letter in each, the order of the above Quasi-Covariant will be the same as its extent. This is the case with the reciprocal function, which is a co-reciprocant (*i. e.* a Quasi-Covariant relative to V).

Ex. $$y^2 \left(e^{\frac{x}{y}}\right) c = cy^2 + Vcxy + \frac{V^2 c}{1.2} x^2 = cy^2 + 5abxy + 5a^3x^2.$$

The discriminant of this is the pure reciprocant

$$5a^2\left(ac-\frac{5b^2}{4}\right).$$

As an additional example, consider the pair of linear co-reciprocants

$$4a\,(4ac-5b^2)\,x+(5ad-7bc)\,y,$$
$$50a\,(a^2d-3abc+2b^3)\,x+(25abd-32ac^2+5b^2c)\,y.$$

The resultant of this pair is

$$2a\,(125a^3d^2-750a^2bcd+500ab^3d+256a^2c^3+165ab^2c^2-300b^4c),$$

i. e. is the Quasi-Discriminant multiplied by $2a$.

LECTURE XX.

"Quintessenced into a finer substance."—*Drummond of Hawthornden.*

Before proceeding with the proper subject of this day's lecture, I should like to mention a geometrical theorem which has fallen in my way, and which, *inter alia*, gives an immediate proof of the existence of 27 straight lines on a general cubic surface. It is proved by means of a Lemma (itself of quasi-geometrical origin) which finds its principal application in an extension of Bring's or Tschirnhausen's method, and shows how any number of specified terms, reckoning from either end, can be taken away from any equation of a sufficiently high degree.*

Subjectively speaking, I was led to the Lemma by considering the question, closely connected with Differential Invariants, of the method of depriving a linear differential equation of several terms.

Let ϕ be a cubic and u a linear function in x, y, z, t, say

$$\phi=ax^3+\ldots.+fx^2y+\ldots.,$$
$$u=lx+my+nz+pt.$$

Then, if ψ is a scroll which contains all the straight lines on $\phi+\lambda u^3$, when the parameter λ has any arbitrary numerical value from $+\infty$ to $-\infty$, I prove that

$$\psi=\phi^2A+\phi u^3B+u^6C,$$

*I recover all Hamilton's results contained in his Report to the British Association, 1836, "On Jerrard's Method," in a much more clear and concise manner, and make important additions to his theory.

where $\quad\psi$ is of the degree 15 in the variables x, y, z, t,

. . . . 6 in the coefficients (l, m, n, p) of u,

. . . . 11 . . . . $(a, \ldots\ldots)$ of ϕ.

Or, more briefly, in	x	l	a
ψ is of degree	15	6	11, and consequently
C	9	0	11.

The intersections of ϕ with ψ are its intersections with u^6 and with C, of which the intersections with the arbitrary plane u^6 are clearly foreign to the question, but the cubic ϕ and the $9^c C$ intersect in 27 straight lines, which are the 27 ridges on ϕ.

C is identical with the covariant found by Clebsch and given in Salmon's Geometry of Three Dimensions at the end of the chapter on Cubic Surfaces. It may with propriety be called the Clebschian.

By giving the parameter λ (which occurs in $\phi + \lambda u^3$) an infinitesimal variation, it is easily proved that

$$B = -2EC, \; A = E^2C, \; E^3C = 0,$$

where E is the operator $l^3\partial_a + \ldots . + 3l^2m\partial_f + \ldots .$, which may be simply and completely defined by its property of changing the general cubic ϕ into $(lx + my + nz + pt)^3$.

The equation $E^3C = 0$ expresses a new property of the Clebschian: it shows that if a, f are the coefficients of x^3 and any other term in ϕ containing x^2, neither a^3 nor a^2f can occur in any one of the terms of C. Defining a principal term in ϕ as one which contains the cube of one of the variables, and a term adjacent to it as one which contains the square of the same variable, this is equivalent to saying that neither the cube of the coefficient of a principal term nor its square multiplied by the coefficient of any adjacent term can appear in any of the terms of C.

An interesting special case of the general theorem is when the arbitrary plane u is taken to be one of the planes of reference, say $u = x$. Then

$$l = 1, \; m = 0, \; n = 0, \; p = 0,$$

and the operator E becomes simply $\dfrac{d}{da}$. Thus we learn that

$$\phi^2 \frac{d^2C}{da^2} - 2x^3\phi \frac{dC}{da} + x^6 C$$

is a Scroll of the fifteenth order which contains all the Ridges on

$$\phi + \lambda x^3$$

for any arbitrary value of the parameter λ.

It also contains 6 times over the curve of intersection of $\phi = 0$ with $x = 0$.

I now propose to give the substance, with a brief commentary, of some very interesting letters I have recently received from Capt. MacMahon. I abstain from giving a proof of his results, as I am informed that he intends to do this himself at an early meeting of the London Mathematical Society.

Using V to signify the Reciprocant Annihilator and Ω the Annihilator of Invariants, we have studied the properties of

$$V\frac{d}{dx} - \frac{d}{dx}V$$

and those of
$$\Omega\frac{d}{dx} - \frac{d}{dx}\Omega.$$

These may be written in the form

$$\left|\begin{matrix} V & \frac{d}{dx} \\ V & \frac{d}{dx} \end{matrix}\right| \qquad \left|\begin{matrix} \Omega & \frac{d}{dx} \\ \Omega & \frac{d}{dx} \end{matrix}\right|,$$

and may be called alternants to $V, \frac{d}{dx}$ and to $\Omega, \frac{d}{dx}$ respectively.

It has been shown in Lecture VII (see Vol. VIII, p. 238 of this Journal) that

$$V\frac{d}{dx} - \frac{d}{dx}V = 2(3i + w)a.$$

The corresponding formula is

$$\Omega\frac{d}{dx} - \frac{d}{dx}\Omega = 3i + 2w,$$

as may be seen by writing $\varkappa = 0,\ \lambda = 3,\ \mu = 4,\ \nu = 5, \ldots$ in a more general formula given in Lecture V (p. 224).

Observe that operating with the alternant to $\Omega, \frac{d}{dx}$ is equivalent to multiplication by a number, and that operating with the alternant to $V, \frac{d}{dx}$ merely introduces a numerical multiple of a as a factor. No such property exists for the Alternant
$$V\Omega - \Omega V,$$
but one much more extraordinary.

MacMahon has found that this alternant, which he calls J, is a generator to a Reciprocant and a generator to an Invariant; *i. e.* it converts a Reciprocant into another Reciprocant, and an Invariant into another Invariant. As regards a Differential Invariant, which is at once an Invariant and a Reciprocant, it is an Annihilator. He shows, in fact, that

$$\Omega J - J\Omega = 0$$

and

$$VJ - JV = 0.$$

If, then, $\Omega R = 0$, it follows immediately that $\Omega(JR) = 0$; *i. e.* if R is an invariant, JR is so too. And in like manner, if

$$VR = 0, \quad V(JR) = 0,$$

i. e. if R is a reciprocant, so is JR.

Of course, if M is a Differential Invariant,

$$JM = V(\Omega M) - \Omega(VM) = 0.$$

Let me here give a caution which may be necessary: The fact that a form is annihilated by J is not sufficient to show that it is a Differential Invariant, though all Differential Invariants are necessarily annihilated by J. Forms exist which are subject to annihilation by

$$J = a^2\partial_c + 3ab\partial_d + \ldots\ldots,$$

but are, notwithstanding, *neither* invariants nor reciprocants.

Such a form is the monomial b, which is obviously annihilated by J. Another is $ad - 3bc$. For, since

$$a^2d - 3abc + 2b^3$$

is a Differential Invariant, we have

$$J(a^2d - 3abc + 2b^3) = 0.$$

But $\qquad Jb^3 = 0 \quad \text{and} \quad Ja = 0;$

therefore, also, $\qquad aJ(ad - 3bc) = 0.$

The general theorem is as follows, and is a most remarkable one: If we write

$$\begin{aligned} mP(m, \mu, v, n) = \mu a^m \partial_{a_n} &+ (\mu + v)\, ma^{m-1}b\partial_{a_{n+1}} \\ &+ (\mu + 2v)\left(ma^{m-1}c + \frac{m(m-1)}{2}a^{m-2}b^2\right)\partial_{a_{n+2}} \\ &+ (\mu + 3v)\left\{ma^{m-1}d + m(m-1)a^{m-2}bc \right.\\ &\left. + \frac{m(m-1)(m-2)}{6}a^{m-3}b^3\right\}\partial_{a_{n+3}} + \ldots\ldots, \end{aligned}$$

where the coefficients of the terms inside the brackets are the same as those of the corresponding terms in the expansion of $(a+b+c+\ldots.)^m$, and where a_n stands for the n^{th} letter of the series $a, b, c, d, \ldots.$, then Capt. MacMahon establishes that *the alternant of any two P's is another P*.

A question here suggests itself naturally: What would be the alternant of three or more P's? For instance, would the alternant

$$\begin{vmatrix} P_1 & P_2 & P_3 \\ P_1 & P_2 & P_3 \\ P_1 & P_2 & P_3 \end{vmatrix} = P_1P_2P_3 - P_1P_3P_2 + P_2P_3P_1 - P_2P_1P_3 + P_3P_1P_2 - P_3P_2P_1$$

be another P?*

Moreover, he obtains expressions for the parameters m, μ, v, n of the resulting P in terms of the parameters of its two components. He proves that if P_1, P_2 are the two components whose alternant is P, supposing

m_1, μ_1, v_1, n_1 to be the parameters of P_1,
$m_2, \mu_2, v_2, n_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots P_2$,

then the parameters m, μ, v, n of their resultant P are given by the equations

$$m = m_1 + m_2 - 1,$$
$$\mu = (m_1 + m_2 - 1)\left\{\frac{\mu_2}{m_2}(\mu_1 + n_2v_1) - \frac{\mu_1}{m_1}(\mu_2 + n_1v_2)\right\},$$
$$v = (n_2 - n_1)v_1v_2 - \frac{m_2 - 1}{m_1}\mu_1v_2 + \frac{m_1 - 1}{m_2}\mu_2v_1,$$
$$n = n_1 + n_2.$$

It will be seen that Ω and V are special forms of P. Thus,

$$\Omega = P(1, 1, 1, 1),$$
$$V = P(2, 4, 1, 1).$$

Now, if the second and third parameters are zero, every term of P vanishes, and MacMahon finds that in the following two cases the second and third parameters of the resultant above given vanish.

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*In my Multiple Algebra investigations, which I hope some day to resume, I have made important use of similar Alternants, which, it may be noticed, do not vanish when their elements are non-commutative. In this connection it is well worthy of observation that the P's (as indeed would be true of any operators linear in the differential inverses) obey the *associative* law.

It would be interesting to ascertain under what arithmetical conditions, if any, other than MacMahon's, *any* two linear operators of the same general form as his P's become commutative.

Perhaps it would also be worthy of inquiry whether the P theory might not admit of extension in some form to operators non-linear in the differential inverses, and whether to every such operator of degrees i and j in the letters and their differential inverses there is not correlated another in which i and j are interchanged.

(1). Supposing $\frac{\mu_1}{m_1v_1}$ to be an integer, this takes place when the two component system of parameters are

$$\begin{array}{llll} m_1, & \mu_1, & v_1, & n_1, \\ m_2, & \mu_1 m_2, & m_1v_1, & n_1 + \frac{\mu_1}{m_1v_1}(m_2 - m_1). \end{array}$$

(2). When they are

$$\begin{array}{llll} m_1, & \mu_1, & v_1, & n_1, \\ m_2, & n_1 m_2, & m_1 - 1, & \frac{\mu_1}{m_1v_1}(m_2 - 1). \end{array}$$

Now,
$$P(1, 1, 1, 1) = \Omega,$$
$$P(2, 4, 1, 1) = V,$$

and by the law of composition

$$J = \Omega V - V\Omega = P(2, 2, 1, 2).$$

Also, $\left.\begin{array}{l} 2, 2, 1, 2 \\ 1, 1, 1, 1 \end{array}\right\}$ will be found to come under the first case;

and $\left.\begin{array}{l} 2, 2, 1, 2 \\ 2, 4, 1, 1 \end{array}\right\}$ . . . . the second.

Hence, $\Omega J - J\Omega = 0$ and $VJ - JV = 0$.

The above theorem is one of extraordinary beauty, and must play an important part in the future of Algebra.

In another letter Capt. MacMahon calls my attention to the fact that the operator called by me Cayley's generator P, in Lecture IV of this course (*American Journal of Mathematics*, Vol. VIII, p. 221), is a particular case of one of a much more general character given by him in the Quarterly Mathematical Journal (Vol. XX, p. 362).

He also states that every pure reciprocant, when multiplied by the needful power of a, is an invariant of the binary quantic

$$\begin{aligned} &\{2.(2n+1)!\}\, a^{n+1} - n\,\{1!\,(2n+1)!\}\, a^{n-1}bt \\ &\qquad + \frac{n(n-1)}{1.2}\{2!\,(2n)!\}\left\{a^{n-2}c + \frac{n-2}{2}a^{n-3}b^2\right\}t^2 \\ &- \frac{n(n-1)(n-2)}{1.2.3}\{3!\,(2n-1)!\}\left\{a^{n-3}d + (n-3)\,a^{n-4}bc + \frac{(n-3)(n-4)}{1.2.3}a^{n-5}b^3\right\}t^3 \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

which I have written in the non-homogeneous form.

But this expression is (to a numerical factor près) identical with the numerator of $\frac{d^{n+2}x}{dy^{n+2}}$ when t, a, b, ... are taken to be the modified differential derivatives $\frac{dy}{dx}$, $\frac{1}{2}\frac{d^2y}{dx^2}$, $\frac{1}{2.3}\frac{d^3y}{dx^3}$, ... See my note on Burman's law for the Inversion of the Independent Variable (Supplement to the Philosophical Magazine for December, 1854).

The property that its invariants are pure reciprocants has already been proved in the lectures.

LECTURE XXI.

I take blame to myself for not earlier communicating to the class the substance of a note of Mr. Hammond's under date of January 20th, 1886, in which he makes an interesting application of the theorem that any invariant of the form

$$y^n \left(e^{\frac{x}{y}V}\right) F(a, b, c, \dots.),$$

in which the function F is subject to the condition

$$V^{n+1}F = 0,$$

or of any combination of such forms, is a pure reciprocant.

Forms such as the above, whose invariants are pure reciprocants, he calls *co-reciprocants.* It follows that any covariant of one or more co-reciprocants is itself a co-reciprocant, for any invariant of a covariant is an invariant.

Taking F to be a single letter b, c, d, he forms the functions

(1) $$by + 2a^2x,$$

(2) $$cy^2 + 5abxy + 5a^3x^2,$$

(3) $$dy^3 + 3(2ac + b^2)xy^2 + 21a^2bx^2y + 14a^4x^3,$$

in which

$$2a^2 = Vb,$$

$$5ab = Vc, \quad 5a^3 = \frac{V^2c}{1.2},$$

$$3(2ac + b^2) = Vd, \quad 21a^2b = \frac{V^2d}{1.2}, \quad 14a^4 = \frac{V^3d}{1.2.3}.$$

On writing $y = t$, $x = -1$, it will be observed that these three forms are the numerators of

$$\frac{1}{3!}\frac{d^3x}{dy^3},\ \frac{1}{4!}\frac{d^4x}{dy^4},\ \frac{1}{5!}\frac{d^5x}{dy^5}.$$

The Jacobian of (1) and (2) is

$$(4ac - 5b^2)\,ay;$$

the coefficient of ay is the familiar pure reciprocant $4ac - 5b^2$.

The Jacobian of (1) and (3) is the determinant

$$\begin{vmatrix} b & 2a^2 \\ dy^2 + (4ac - 5b^2)\,xy & (2ac + b^2)\,y^2 \end{vmatrix},$$

which is divisible by y, giving the quotient

(4) $$(2a^2d - 2abc - b^3)\,y + 2a^2(4ac - 5b^2)\,x.$$

This is

$$y\left(e^{\frac{x}{y}V}\right)(2a^2d - 2abc - b^3),$$

the terms involving $\frac{x^2}{y}$, $\frac{x^3}{y^2}$, ... vanishing identically.

Looking at $2a^2d - 2abc - b^3$ as the anti-source to a Co-reciprocant,* we might at first sight expect that it would give rise to a co-reciprocant of the third order in x, y, whereas we see it is the anti-source of a linear co-reciprocant.

We have $$V(2a^2d - 2abc - b^3) = 2a^2(4ac - 5b^2).$$

Combining this with

$$V(a^2d - 3abc + 2b^3) = 0 \quad \text{(the well-known Mongian)},$$

and dividing by a, he obtains

$$V(5ad - 7bc) = 4a(4ac - 5b^2).$$

Hence

(5) $$(5ad - 7bc)\,y + 4a(4ac - 5b^2)\,x$$

is a cò-reciprocant. It is in fact (4) reduced in degree.

The Jacobian of (5) and of $cy^2 + 5abxy + 5a^3x^2$, *i. e.*

$$\begin{vmatrix} 5ad - 7bc & 4a(4ac - 5b^2) \\ 2cy + 5abx & 5aby + 10a^3x \end{vmatrix},$$

* What differentiates Reciprocants from Invariants is that we have no reverser to V as O is to Ω in the theory of Invariants, *i. e.* no reverser which does not introduce an additional letter.

The coefficients of a covariant are obtained either from the source by continually operating with O, or from the anti-source by continually operating with Ω. But in the case of a co-reciprocant, we are only able to proceed in one direction (viz. from the anti-source, or coefficient of the highest power of y, to the source), as we have only one operator, V, at our disposal.

will divide by a, and gives the new linear co-reciprocant

(6) $$(25abd - 32ac^2 + 5b^2c)\,y + 50a\,(a^2d - 3abc + 2b^3)\,x.$$

The coefficient of y is of weight 4, but instead of giving rise to a co-reciprocant of the 4th order, we see that this again is the anti-source of a linear co-reciprocant.

The resultant of the two linear co-reciprocants (4) and (6) divided by a numerical multiple of a gives the well-known Quasi-Discriminant $125a^3d^2 + \ldots\ldots$, as was stated at the end of Lecture XIX.

The noticeable fact is that (including $by + 2a^2x$) there exist 3 linear independent co-reciprocants of extent 3. Probably there are no more, but this requires proof.

The promised land of Differential Invariants or Projective Reciprocants is now in sight, and the remainder of the course will be devoted to its elucidation. Twenty lectures have been given on the underlying matter, and probably ten more, at least, will have to be expended on this higher portion of the theory.

One is surprised to reflect on the change which has come over the face of Algebra in the last quarter of a century. It is now possible to enlarge to an almost unlimited extent on any branch of it. These thirty lectures, embracing only a fragment of the theory of reciprocants, might be compared to an unfinished epic in thirty cantos. Does it not seem as if Algebra had attained to the character of a fine art, in which the workman has a free hand to develop his conceptions as in a musical theme or a subject for painting? Formerly it consisted almost exclusively of detached theorems, but now-a-days it has reached a point in which every properly developed algebraical composition, like a skilful landscape, is expected to suggest the notion of an infinite distance lying beyond the limits of the canvas.

It is quite conceivable that the results we have been investigating may be descended upon from a higher and more general point of view. Many circumstances point to such a consummation being probable. But man must creep before he can walk or run, and a house cannot be built downwards from the roof. I think the mere fact that our work enables us to simplify and extend the results obtained by so splendid a genius as M. Halphen, is sufficient to convey to us the assurance that we have not been beating the wind or chasing a phantom, but doing solid work. Let me instance one single point: M. Halphen has

succeeded, by a prodigious effort of ingenuity, in obtaining the differential equation to a cubic curve with a given absolute invariant. His method involves the integration of a complicated differential equation. In the method which I employ the same result is obtained by a simple act of substitution in an exceedingly simple special form of Aronhold's S and T, capable of being executed in the course of a few minutes on half a sheet of paper, without performing any integration whatever. This will be seen to be a simple inference from the theorem invoked under three names, to which allusion has been made in a preceding lecture and the demonstration of which will shortly occupy our attention.

Before entering upon the theory of Differential Invariants, I think it desirable to bring forward the exceedingly valuable and interesting communication with which I have been favored by M. Halphen establishing *à priori* the existence of *invariants* in general.

SUR L'EXISTENCE DES INVARIANTS.

(*Extracted from a Letter of M. Halphen to Professor Sylvester.*)

Dans des théories diverses on a rencontré des Invariants sans qu'on ait pénétré la cause générale de leur existence. C'est cette lacune qu'il s'agit ici de faire disparaître.

1. Soient $A, B, \ldots . L$ des quantités auxquelles on puisse attribuer des valeurs *ad libitum.*

Une *substitution* consiste à remplacer ces quantités $(A, B, \ldots . L)$ par d'autres $(a, b, \ldots . l)$.

Les substitutions, que l'on doit considérer ici, sont définies par des relations algébriques, de forme supposée donnée, mais contenant des *paramètres* arbitraires $p, q, \ldots .$

$$(1) \qquad \begin{cases} a = f(A, B, \ldots . L;\, p, q, \ldots .), \\ b = f_1(A, B, \ldots . L;\, p, q, \ldots .), \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

Soit maintenant une seconde substitution, de même espèce, mais avec d'autres paramètres $\pi, \chi, \ldots .$, et donnant lieu à $(\alpha, \beta, \ldots . \lambda)$, en sorte qu'on ait

$$(1 \text{ bis}) \qquad \begin{cases} \alpha = f(A, B, \ldots . L;\, \pi, \chi, \ldots .), \\ \beta = f_1(A, B, \ldots . L;\, \pi, \chi, \ldots .), \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

2. DÉFINITION. Les substitutions dont il s'agit forment un GROUPE, si, quels que soient les paramètres $p, q, \ldots\ldots, \pi, \chi, \ldots\ldots$, ainsi que $A, B, \ldots\ldots L$, il existe des quantités $P, Q, \ldots\ldots$ vérifiant les égalités semblables

$$\text{(1 ter)} \qquad \begin{cases} \alpha = f(a, b, \ldots\ldots l;\ P, Q, \ldots\ldots), \\ \beta = f_1(a, b, \ldots\ldots l;\ P, Q, \ldots\ldots), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

Les invariants sont l'apanage exclusif des substitutions formant groupe. On va le montrer. Mais auparavant, pour éviter toute confusion, on doit faire une remarque sur la définition.

3. Dans les diverse théories où l'on a rencontré des Invariants, les substitutions forment groupe, en effet, suivant cette definition; mais il s'y rencontre encore une circonstance particulière de plus, c'est que les paramètres $P, Q, \ldots\ldots$ de la substitution composée (1 ter) dépendent uniquement des paramètres $p, q, \ldots\ldots, \pi, \chi, \ldots\ldots$ des substitutions composantes, (1) et (1 bis). Cette propriété *n'est pas nécessaire* à l'existence des Invariants, et nous ne la supposerons pas ici. Il sera donc entendu que $P, Q, \ldots\ldots$ peuvent dépendre, non seulement de $p, q, \ldots\ldots, \pi, \chi, \ldots\ldots$, mais aussi de $A, B, \ldots\ldots L$.

EXEMPLES:

I.
$$\begin{aligned} a &= Ap^2, & b &= Apq + Bp, & c &= Aq^2 + 2Bq + C; \\ \alpha &= A\pi^2, & \beta &= A\pi\chi + B\pi, & \gamma &= A\chi^2 + 2B\chi + C; \\ \alpha &= aP^2, & \beta &= aPQ + bP, & \gamma &= aQ^2 + 2BQ + C; \end{aligned}$$
$$P = \frac{\pi}{p}, \qquad Q = \frac{\chi - q}{p}.$$

P et Q ne dépendent pas de A, B, C.

II.
$$\begin{aligned} a &= A^3p^2, & b &= A^2pq + ABp, & c &= Aq^2 + 2Bq + C; \\ \alpha &= A^3\pi^2, & \beta &= A^2\pi\chi + AB\pi, & \gamma &= A\chi^2 + 2B\chi + C; \\ \alpha &= a^3P^2, & \beta &= a^2PQ + abP, & \gamma &= aQ^2 + 2bQ + c; \end{aligned}$$
$$P = \frac{\pi}{A^3p^3}, \qquad Q = \frac{\chi - q}{Ap}.$$

P et Q dépendent de A.

Dans ces deux exemples, il y a un invariant absolu, $\dfrac{B^2 - AC}{A}$.

4. Dans la substitution (1) nous supposerons que le nombre des paramètres soit inférieur au nombre des quantités $A, B, \ldots\ldots L$.

Soient ainsi m le nombre des paramètres $p, q, \ldots.,$
n le nombre des quantités $A, B, \ldots. L,$

on suppose
$$m < n.$$

Cela étant, on peut éliminer les paramètres entre les équations (1), et il reste $(n - m)$ équations

$$(2) \qquad \begin{cases} F\,(a, b, \ldots. l;\ A, B, \ldots. L) = 0, \\ F_1(a, b, \ldots. l;\ A, B, \ldots. L) = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

THÉORÈME: *Si les substitutions considérées forment* GROUPE, *les* $(n - m)$ *équations* (2) *peuvent être mises sous la forme*

$$(3) \qquad \begin{cases} \Phi\,(a, b, \ldots. l) = \Phi\,(A, B, \ldots. L), \\ \Phi_1(a, b, \ldots. l) = \Phi_1(A, B, \ldots. L), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

en d'autres termes, *il y a* $(n - m)$ *invariants absolus.*

Réciproquement, s'il y a $(n - m)$ *invariants absolus* (distincts), *les substitutions forment groupe.*

5. DÉMONSTRATION. Prouvons d'abord la seconde partie, ou réciproque. Voici l'hypothèse: des équations (1), par élimination de $p, q, \ldots.$ resultent les équations (3).

Par conséquent, $A, B, \ldots. L$ et $a, b, \ldots. l$ étant quelconques, mais satisfaisant aux équations (3), on peut déterminer $p, q,$ au moyen des équations (1).

Soient $A, B, \ldots. L,\ p, q, \ldots.,\ \pi, \chi, \ldots.$ pris arbitrairement, et $a, b, \ldots. l,\ \alpha, \beta, \ldots. \lambda$ déterminés par (1) et (1 bis). Suivant l'hypothèse, on a

$$\Phi(a, b, \ldots. l) = \Phi(A, B, \ldots. L) \text{ et } \Phi(\alpha, \beta, \ldots. \lambda) = \Phi(A, B, \ldots. L);$$

donc
$$\Phi(a, b, \ldots. l) = \Phi(\alpha, \beta, \ldots. \lambda), \text{ etc.}$$

Donc on peut déterminer $P, Q, \ldots.$ par les équations (1 ter), ce qu'il fallait démontrer.

Démontrons maintenant la première partie, ou théorème direct. Par hypothèse, $A, B, \ldots. L,\ p, q, \ldots.,\ \pi, \chi, \ldots.$ étant pris à volonté et $a, b, \ldots. l,\ \alpha, \beta, \ldots. \lambda$ déterminés au moyen de (1) et (1 bis), il en résulte les relations (1 ter).

Des équations (1) résulte le système (2); de même, de (1 bis) et de (1 ter) resultent

$$\text{(2 bis)} \quad \begin{cases} F\,(\alpha, \beta, \ldots. \lambda;\ A, B, \ldots. L) = 0, \\ F_1(\alpha, \beta, \ldots. \lambda;\ A, B, \ldots. L) = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

$$\text{(2 ter)} \quad \begin{cases} F\,(\alpha, \beta, \ldots. \lambda;\ a, b, \ldots. l) \quad = 0, \\ F_1(\alpha, \beta, \ldots. \lambda;\ a, b, \ldots. l) \quad = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

Je dis que le système (2 ter) résulte de (2) et de (2 bis).

En effet, $a, b, \ldots. l$ et $\alpha, \beta, \ldots. \lambda$ n'étant definis que par (1) et (1 bis), le système (2 ter) résulte de (1) et de (1 bis) par l'élimination de $p, q, \ldots.,$ $\pi, \chi, \ldots.$ et $A, B, \ldots. L$. Mais l'élimination de $p, q, \ldots.$ remplace le système (1) par le système (2), celle de $\pi, \chi, \ldots.$ remplace le système (1 bis) par (2 bis); donc (2 ter) résulte de l'élimination de $A, B, \ldots. L$ entre (2) et (2 bis).

Le système (2), (2 bis) est formé par $2(n-m)$ équations, et cependant l'élimination de n lettres $A, B, \ldots. L$, au lieu de donner $(n-2m)$ équations, en donne $(n-m)$, les équations (2 ter). Si donc on élimine seulement $(n-m)$ lettres $A, B, \ldots. G$, les m autres $H, \ldots. L$ disparaîtront d'elles-mêmes. Tirons $A, B, \ldots. G$ des équations (2), et nous aurons

$$\begin{aligned} A &= \Psi\,(a, b, \ldots. l;\ H, \ldots. L), \\ B &= \Psi_1(a, b, \ldots. l;\ H, \ldots. L), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Tirons de même $A, B, \ldots. G$ des équations (2 bis), et nous aurons

$$\begin{aligned} A &= \Psi\,(\alpha, \beta, \ldots. \lambda;\ H, \ldots. L), \\ B &= \Psi_1(\alpha, \beta, \ldots. \lambda;\ H, \ldots. L), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Le résultat de l'élimination est donc représenté par $(n-m)$ équations telles que

$$\text{(4)} \quad \begin{cases} \Psi\,(a, b, \ldots. l;\ H, \ldots. L) = \Psi\,(\alpha, \beta, \ldots. \lambda;\ H, \ldots. L), \\ \Psi_1(a, b, \ldots. l;\ H, \ldots. L) = \Psi_1(\alpha, \beta, \ldots. \lambda;\ H, \ldots. L), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

et l'on sait que $H, \ldots. L$ disparaissent, d'eux-mêmes, de ces équations.

En assignant donc à $H, \ldots. L$ des valeurs numériques à volonté, on voit donc bien que les équations résultants, équivalentes à (2 ter), ont la forme

$$\begin{aligned} \Phi\,(a, b, \ldots. l) &= \Phi\,(\alpha, \beta, \ldots. \lambda), \\ \Phi_1(a, b, \ldots. l) &= \Phi_1(\alpha, \beta, \ldots. \lambda), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

C'est ce qu'il fallait démontrer.

6. REMARQUES. Si les équations (4) sont rationnelles, la disparition de $H, \ldots . L$ exige que Ψ ait la forme suivante

$$\Psi = \Phi(a, b, \ldots . l)\,\Theta(H, \ldots . L) + \theta(H, \ldots . L),$$

et de même pour Ψ_1, etc. Sous cette forme, on voit que Θ et θ disparaissent dans les équations (4), et l'invariant résultant est Φ.

Mais, si les équations (4) sont irrationnelles, la disparition de $H, \ldots . L$ peut n'être pas immédiate. En assignant à $H, \ldots . L$ des valeurs numériques à volonté, comme on l'a dit dans la démonstration, c'est-à-dire en considérant $H, \ldots . L$ comme des *constantes arbitraires*, on voit les invariants se présenter avec des constantes arbitraires. Ceci ne doit pas étonner, puisqu'il s'agit ici d'invariants *absolus*, que l'on peut effectivement modifier en leur ajoutant des constantes arbitraires ou en les multipliant par des constantes arbitraires, sans troubler la propriété d'invariance.

L'analyse employée dans la démonstration fournit un moyen régulier de former les invariants; ce moyen consiste à éliminer les paramètres dans les équations (1), puis à résoudre par rapport à $(n-m)$ quantités $A, B, \ldots . G$. Mais, les substitutions forment groupe, on peut aussi résoudre par rapport à $a, b, \ldots . g$, en éliminant les paramètres.

EXEMPLE: $\quad a = Ap^2,\ b = Apq + Bp,\ c = Aq^2 + 2Bq + C.$

En résolvant par rapport à c, c'est-à-dire en tirant p, q des deux premières, on obtient

$$c = A\left(\frac{b - Bp}{Ap}\right)^2 + 2B\,\frac{b - Bp}{Ap} + C = \frac{b^2}{Ap^2} + C - \frac{B^2}{A} = \frac{b^2}{a} + C - \frac{B^2}{A}.$$

Voici l'invariant $C - \dfrac{B^2}{A}$.

En résolvant par rapport à b, on trouve $b = \sqrt{a}\sqrt{\dfrac{B^2 - AC}{A}} + c$, ce qui donne l'invariant $\dfrac{B^2 - AC}{A} + c$, où c est une constante arbitraire.

LECTURE XXII.

E pur si muove.

The theory still moves on. We have now emerged from the narrows and are entering on the mid-ocean of Differential Invariants, or of Principiants, as I have called them. These, it will now be seen, are perfectly defined by their property of being at one and the same time invariants and pure reciprocants. In other words, if P be a Principiant, it has both Ω and V for its annihilators. Thus, *ex. gr.*, the Mongian

$$A = a^2d - 3abc + 2b^3$$

is necessarily a Principiant. For

$$\Omega A = (a\partial_b + 2b\partial_c + 3c\partial_d)(a^2d - 3abc + 2b^3) = 0,$$

and at the same time

$$VA = \{2a^2\partial_b + 5ab\partial_c + (6ac + 3b^2)\partial_d\}(a^2d - 3abc + 2b^3) = 0.$$

Among Pure Reciprocants, those only are entitled to rank as Principiants whose form is persistent (merely taking up an extraneous factor, but otherwise unchanged) under the most general homographic substitution (see Lecture XIII, *American Journal of Mathematics*, Vol. IX, p. 17). We have therefore to show that such reciprocants and no others are subject to annihilation by Ω.

With this end in view, let us consider the effect of substituting $\frac{x}{1+hx}$ for x and $\frac{y}{1+hx}$ for y in any rational integral function of y and its derivatives with respect to x. Suppose that, in consequence of this substitution, the function

$$F(y, y_1, y_2, y_3, \ldots . y_n)$$

becomes changed into

$$F_1(x, y, y_1, y_2, y_3, \ldots . y_n);$$

then the transformed function will be

$$F(Y, Y_1, Y_2, Y_3, \ldots . Y_n),$$

where $X = \frac{x}{1+hx}$, $Y = \frac{y}{1+hx}$, and $Y_1, Y_2, Y_3, \ldots . Y_n$ are the successive derivatives of Y with respect to X.

If, for the moment, we agree to consider h as an infinitesimal (we shall

afterwards give it a finite value), neglecting squares and higher powers of h, we may write

$$X = x - hx^2,$$
$$Y = y - hxy.$$

Hence, by n successive differentiations of Y with respect to X, neglecting squares of h whenever they occur, we deduce

$$\begin{aligned} Y_1 &= y_1 + hxy_1 - hy, \\ Y_2 &= y_2 + 3hxy_2, \\ Y_3 &= y_3 + 5hxy_3 + 3hy_2, \\ Y_4 &= y_4 + 7hxy_4 + 8hy_3, \\ Y_5 &= y_5 + 9hxy_5 + 15hy_4, \\ &\cdots\cdots\cdots\cdots \\ Y_{n-1} &= y_{n-1} + (2n-3)\, hxy_{n-1} + (n-1)(n-3)\, hy_{n-2}, \\ Y_n &= y_n + (2n-1)\, hxy_n + n(n-2)\, hy_{n-1}. \end{aligned}$$

The last of these, for instance, is obtained as follows:

We have $\quad Y_n = \dfrac{dY_{n-1}}{dX}$.

But
$$\frac{d}{dX} = \frac{1}{1-2hx} \cdot \frac{d}{dx} = (1+2hx)\frac{d}{dx},$$
and
$$\begin{aligned} \frac{dY_{n-1}}{dx} &= \frac{d}{dx}\{y_{n-1} + (2n-3)\, hxy_{n-1} + (n-1)(n-3)\, hy_{n-2}\} \\ &= y_n + (2n-3)\, hxy_n + n(n-2)\, hy_{n-1}. \end{aligned}$$

Consequently,
$$\begin{aligned} Y_n &= (1+2hx)\frac{dY_{n-1}}{dx} \\ &= (1+2hx)\{y_n + (2n-3)\, hxy_n + n(n-2)\, hy_{n-1}\} \\ &= y_n + (2n-1)\, hxy_n + n(n-2)\, hy_{n-1}. \end{aligned}$$

On substituting the above values of $Y, Y_1, Y_2, \ldots\ldots Y_n$ in the transformed function, we find immediately

$$F(Y, Y_1, Y_2, \ldots\ldots Y_n) = (1 + hx\nu + h\Theta)\, F(y, y_1, y_2, \ldots\ldots y_n),$$

where ν and Θ are the partial differential operators

$$\begin{aligned} \nu &= -y\partial_y + y_1\partial_{y_1} + 3y_2\partial_{y_2} + 5y_3\partial_{y_3} + 7y_4\partial_{y_4} + \ldots\ldots, \\ \Theta &= -y\partial_{y_1} + 3y_2\partial_{y_3} + 8y_3\partial_{y_4} + 15y_4\partial_{y_5} + \ldots\ldots + n(n-2)\, y_{n-1}\partial_{y_n}. \end{aligned}$$

Changing to our usual notation, we write

$$y_1 = t,\ y_2 = 2a,\ y_3 = 2.3b,\ y_4 = 2.3.4c, \ldots\ldots,$$

and then if F_1 is what F (a rational integral function of $a, b, c, \ldots.$) becomes when we substitute $\frac{x}{1+hx}$, $\frac{y}{1+hx}$ for x, y (regarding h as *infinitesimal*), we have

$$F_1 = (1 + hx\nu + h\Theta)\,F,$$

where $\quad \nu = -y\partial_y + t\partial_t + 3a\partial_a + 5b\partial_b + 7c\partial_c + 9d\partial_d + \ldots.,$

and $\quad \Theta = -y\partial_t + a\partial_b + 2b\partial_c + 3c\partial_d + 4d\partial_e + \ldots.$

In general ν is merely the partial differential operator written above; but when its subject, F, is homogeneous, of degree i, and isobaric, of weight w, in

the letters	$y,$	$t,$	$a,$	$b,$	$c,$	$d, \ldots.$	supposed to be
of degrees	$1,$	$1,$	$1,$	$1,$	$1,$	$1, \ldots.$	
and of weights	$-2,$	$-1,$	$0,$	$1,$	$2,$	$3, \ldots.,$	

its operation is equivalent to multiplication by the number $3i + 2w$. For in this case we have

$$y\partial_y + t\partial_t + a\partial_a + b\partial_b + c\partial_c + d\partial_d + \ldots. = i,$$

and $$-2y\partial_y - t\partial_t \qquad + b\partial_b + 2c\partial_c + 3d\partial_d + \ldots. = w;$$

so that we may regard ν as a number, simply writing

$$\nu = 3i + 2w$$

when we have occasion to do so.

We are now able to show that if F is a persistent form, we must necessarily have $$\Theta F = 0.$$

For $$\frac{F_1}{F} = 1 + \nu hx + \frac{h\Theta F}{F};$$

and consequently, if F_1 is divisible by F (this is what is meant by saying that F is a persistent form), unless ΘF vanishes, $\frac{\Theta F}{F}$ must be a rational integral function of $y, t, a, b, c, \ldots.$ But since the operation of Θ diminishes the weight by unity without altering the degree, $\frac{\Theta F}{F}$ must be of degree 0 and weight -1. The impossibility of the existence of such a function leads to the necessary conclusion that $$\Theta F = 0.$$

Let us apply this result to the case of a pure reciprocant. We have

$$\Theta = -y\partial_t + a\partial_b + 2b\partial_c + 3c\partial_d + \ldots. = -y\partial_t + \Omega.$$

Thus when F is a pure reciprocant, or indeed any function in which t does not appear, $y\partial_t F = 0$ and Θ reduces to Ω. We have therefore shown, in what precedes, that the condition $$\Omega F = 0$$

is necessary to ensure the persistence of the form of F under a particular homographic substitution; *à fortiori*, this condition is also necessarily satisfied when the form of F is persistent under the most general homographic substitution $\left(\text{in which } x,\, y \text{ are changed into } \dfrac{lx+my+n}{l''x+m''y+n''},\ \dfrac{l'x+m'y+n'}{l''x+m''y+n''}\right)$.

The satisfaction of $\Omega F=0$ is of itself inadequate to ensure persistence under the general homographic substitution; the necessary and sufficient condition of pure reciprocants

$$VF=0$$

must also be satisfied. This follows from the fact that the general linear substitution, for which all pure reciprocants are persistent, is merely a particular case of the most general homographic substitution.

It only remains to be proved that the two conditions $VF=0$, $\Omega F=0$, taken conjointly, are sufficient as well as necessary.

In what follows I use a method which may be termed that of composition of variations. Its nature and value will be better understood if I first apply it to the rigorous demonstration of the theorem that the substitution of $x+hy$ for x in the Quantic

$$(a,\, b,\, c,\, \ldots \between x,\, y)^n$$

changes any function whatever of its coefficients, say

$$F(a,\, b,\, c,\, \ldots),\ \text{into}\ e^{h\Omega}F(a,\, b,\, c,\, \ldots).$$

This is not proved, but only verified up to terms of the second order of differentiation, in Salmon's Modern Higher Algebra (3d ed. 1876, p. 59). Remembering that, whatever the order n of the Quantic may be, the changed values of the coefficients $a,\, b,\, c,\, d,\, \ldots$ are

$$\begin{aligned} a' &= a, \\ b' &= b+ah, \\ c' &= c+2bh+ah^2, \\ d' &= d+3ch+3bh^2+ah^3, \\ &\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

what we have to prove is that, for all values of h,

$$F(a',\, b',\, c',\, d',\, \ldots)=e^{h\Omega}F(a,\, b,\, c,\, d,\, \ldots).$$

In other words, if for brevity we write

$$F(a,\, b,\, c,\, \ldots)=F,$$

and

$$F(a',\, b',\, c',\, \ldots)=F_1,$$

it is required to show that

$$F_1 = F + h\Omega F + \frac{h^2}{1.2}\Omega^2 F + \frac{h^3}{1.2.3}\Omega^3 F + \ldots,$$

where $\qquad \Omega = a\partial_b + 2b\partial_c + 3c\partial_d + \ldots.$

When h is infinitesimal, it is obvious that

$$F_1 = F + h\Omega F.$$

Hence, when h has a general value, we may assume

$$F_1 = F + h\Omega F + \frac{h^2}{1.2}P + \frac{h^3}{1.2.3}Q + \frac{h^4}{1.2.3.4}R + \ldots.$$

Let h be increased by the infinitesimal quantity ε; then, considering this increase as resulting from a second substitution similar to the first, we see that F_1 becomes $\qquad F_1 + \varepsilon\Omega F_1$.

But it also becomes

$$F + (h+\varepsilon)\Omega F + \frac{(h+\varepsilon)^2}{1.2}P + \frac{(h+\varepsilon)^3}{1.2.3}Q + \ldots = F_1 + \varepsilon\frac{dF_1}{dh}$$

$$= F_1 + \varepsilon\left(\Omega F + hP + \frac{h^2}{1.2}Q + \frac{h^3}{1.2.3}R + \ldots\right).$$

Equating this to $F_1 + \varepsilon\Omega F_1$, we obtain

$$\Omega F_1 = \Omega F + hP + \frac{h^2}{1.2}Q + \frac{h^3}{1.2.3}R + \ldots.$$

But $\qquad \Omega F_1 = \Omega\left(F + h\Omega F + \frac{h^2}{1.2}P + \frac{h^3}{1.2.3}Q + \ldots\right).$

The comparison of these two expressions gives

$$P = \Omega^2 F,$$
$$Q = \Omega P = \Omega^3 F,$$
$$R = \Omega Q = \Omega^4 F.$$
$$\cdots\cdots\cdots\cdots$$

Substituting these values in the assumed expansion for F_1, there results

$$F_1 = F + h\Omega F + \frac{h^2}{1.2}\Omega^2 F + \frac{h^3}{1.2.3}\Omega^3 F + \ldots,$$

which is the expanded form of

$$F_1 = e^{h\Omega}F.$$

A similar method of procedure will enable us to establish the corresponding but more elaborate formula

$$F_1 = (1 + hx)^{\nu} e^{\frac{h\Theta}{1+hx}}F,$$

in which F is any *homogeneous* and *isobaric* function* of degree i and weight w in y and its modified derivatives ($t, a, b, c, \ldots.$) with respect to x; the operator $\Theta = -y\partial_t + a\partial_b + 2b\partial_c + 3c\partial_d + \ldots.$; the function F_1 is what F becomes in consequence of the substitution of $\frac{x}{1+hx}, \frac{y}{1+hx}$ for x, y; h is any finite quantity, and $\nu = 3i + 2w$.

Before giving the proof of this theorem, I will show that, upon the assumption of its truth, two inverse finite substitutions will, as they ought, nullify each other, leaving the function operated upon unaltered in form.

To avoid needless periphrasis, we call the substitution of $\frac{x}{1+hx}, \frac{y}{1+hx}$ for x, y the substitution h.

Either of the two substitutions, $h, -h$, reverses the effect of the other; for the substitution $-h$ turns

$$\frac{x}{1+hx} \text{ into } \frac{x}{1-hx} \div 1 + \frac{hx}{1-hx} = x,$$

and

$$\frac{y}{1+hx} \text{ into } \frac{y}{1-hx} \div 1 + \frac{hx}{1-hx} = y.$$

The two substitutions $h, -h$, performed successively on F, ought therefore to leave its value unaltered. But by hypothesis the substitution h converts F into F_1; consequently the substitution $-h$ performed on F_1 ought to change it back again into F.

It must be carefully observed that (since the operation of Θ decreases the weight by unity, leaving the degree unchanged) the weight of $\Theta^\kappa F$ is κ units lower than that of F, whilst the degree is the same for both.

Thus for $\quad F$ we have $3i + 2w = \nu$

and for $\quad \Theta^\kappa F \quad 3i + 2(w-\kappa) = \nu - 2\kappa$.

Hence the substitution $-h$, which changes

$$F \text{ into } (1-hx)^\nu e^{-\frac{h\Theta}{1-hx}} F,$$

also changes

$$\Theta F \quad \text{"} \quad (1-hx)^{\nu-2} e^{-\frac{h\Theta}{1-hx}} \Theta F,$$

$$\Theta^2 F \quad \text{"} \quad (1-hx)^{\nu-4} e^{-\frac{h\Theta}{1-hx}} \Theta^2 F,$$

. . . .

and in general

$$\Theta^\kappa F \text{ into } (1-hx)^{\nu-2\kappa} e^{-\frac{h\Theta}{1-hx}} \Theta^\kappa F.$$

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*F need not be integral or even rational; whenever it is homogeneous or isobaric, ν will be a number.

Moreover, $1+hx$ becomes $1+\dfrac{hx}{1-hx}=(1-hx)^{-1}$, so that

$$\begin{aligned}(1+hx)^{\nu-\kappa}\Theta^\kappa F \text{ becomes } &(1-hx)^{-(\nu-\kappa)}(1-hx)^{\nu-2\kappa}e^{-\frac{h\Theta}{1-hx}}\Theta^\kappa F\\ &=(1-hx)^{-\kappa}e^{-\frac{h\Theta}{1-hx}}\Theta^\kappa F\\ &=e^{-\frac{h\Theta}{1-hx}}(1-hx)^{-\kappa}\Theta^\kappa F \quad \text{(since } \Theta \text{ does not act on } x).\end{aligned}$$

Consequently,

$$\begin{aligned}(1+hx)^\nu F \quad &\text{becomes } e^{-\frac{h\Theta}{1-hx}}F,\\ (1+hx)^{\nu-1}\Theta F \quad &\text{“} \quad e^{-\frac{h\Theta}{1-hx}}(1-hx)^{-1}\Theta F,\\ (1+hx)^{\nu-2}\Theta^2 F \quad &\text{“} \quad e^{-\frac{h\Theta}{1-hx}}(1-hx)^{-2}\Theta^2 F,\\ &\dots\dots\dots\dots\end{aligned}$$

And since, by the formula to be verified,

$$F_1=(1+hx)^\nu F+h(1+hx)^{\nu-1}\Theta F+\frac{h^2}{1.2}(1+hx)^{\nu-2}\Theta^2F+\dots\dots,$$

$$\begin{aligned}F_1 \text{ becomes } &e^{-\frac{h\Theta}{1-hx}}\left\{1+h(1-hx)^{-1}\Theta+\frac{h^2}{1.2}(1-hx)^{-2}\Theta^2+\dots\dots\right\}F\\ &=e^{-\frac{h\Theta}{1-hx}}e^{\frac{h\Theta}{1-hx}}F=F.\end{aligned}$$

LECTURE XXIII.

We now proceed to show how the composition of variations can be made to furnish a strict proof of the formula

$$F_1=(1+hx)^\nu e^{\frac{h\Theta}{1+hx}}F,$$

which was set forth in the preceding lecture.

As before, calling the change of x, y into $\dfrac{x}{1+hx}$, $\dfrac{y}{1+hx}$, the substitution h, it is easy to see that the *product* of two substitutions, h, ε, is the substitution $h+\varepsilon$. For

$$\frac{x}{1+hx}\div 1+\varepsilon\frac{x}{1+hx}=\frac{x}{1+(h+\varepsilon)x},$$

$$\frac{y}{1+hx}\div 1+\varepsilon\frac{x}{1+hx}=\frac{y}{1+(h+\varepsilon)x}.$$

This shows that if

	F_1 is what F becomes on making the substitution h,							
and	F_2 “	“ F_1	“	“	“	“	“	ε,
then	F_2 “	“ F	“	“	“	“	“	$h+\varepsilon$.

Thus we can find two expressions for F_2, the comparison of which will enable us to assign the coefficients of all the powers of h in the expanded values of F_1.

The first two terms of this expansion were obtained, in the preceding lecture, by treating h as an infinitesimal. We may therefore write

$$F_1 = F + h(\nu x + \Theta)F + \frac{h^2}{1.2}N_2 + \frac{h^3}{1.2.3}N_3 + \ldots .$$

Changing h into $h + \varepsilon$, we deduce

$$F_2 = F + (h+\varepsilon)(\nu x + \Theta)F + \frac{(h+\varepsilon)^2}{1.2}N_2 + \frac{(h+\varepsilon)^3}{1.2.3}N_3 + \ldots .$$

For greater simplicity, let ε be an infinitesimal, and write

$$\frac{F_2 - F_1}{\varepsilon} = \Delta F_1.$$

Then $$\Delta F_1 = (\nu x + \Theta)F + hN_2 + \frac{h^2}{1.2}N_3 + \ldots .$$

Now look at each term in the expansion of F_1 and find its increment (*i. e.* its Δ) when x, y undergo the substitution ε. We thus obtain

$$\Delta F_1 = \Delta F + h\Delta(\nu x + \Theta)F + \frac{h^2}{1.2}\Delta N_2 + \frac{h^3}{1.2.3}\Delta N_3 + \ldots .$$

Comparing these two values of ΔF_1, we find

$$N_2 = \Delta(\nu x + \Theta)F,$$
$$N_3 = \Delta N_2,$$
$$N_4 = \Delta N_3,$$
$$\cdots\cdots\cdots$$

and generally $$N_r = \Delta N_{r-1}.$$

These equations are sufficient to determine all the coefficients of F_1; it only remains to show how the operations Δ may be performed.

We have in fact

$$F_1 = F + h\Delta F + \frac{h^2}{1.2}\Delta^2 F + \frac{h^3}{1.2.3}\Delta^3 F + \ldots ,$$

where $$\Delta F = (\nu x + \Theta)F.$$

But we must not from this rashly infer that

$$\Delta^n F = (\nu x + \Theta)^n F.$$

To do so would be tantamount to regarding ν as a constant number, whereas its value depends on the degree and weight of the subject of operation.

This will be clearly seen in the calculation which follows.* We first generalize the formula $\Delta F = (\nu x + \Theta) F$

by making $\Theta^\kappa F$ the operand instead of F.

Then, since i is the degree and $w - \kappa$ the weight of $\Theta^\kappa F$, instead of

$$3i + 2w = \nu,$$

we have
$$3i + 2(w - \kappa) = \nu - 2\kappa.$$

Thus,
$$\Delta\Theta^\kappa F = \{(\nu - 2\kappa) x + \Theta\} \Theta^\kappa F.$$

Again, since
$$\Delta x = \left(\frac{x}{1 + \varepsilon x} - x\right) \div \varepsilon = -x^2,$$

we find

$$\Delta x^\lambda \Theta^\kappa F = \lambda x^{\lambda-1} \Theta^\kappa F . \Delta x + x^\lambda \Delta\Theta^\kappa F = -\lambda x^{\lambda+1}\Theta^\kappa F + x^\lambda \{(\nu - 2\kappa) x + \Theta\} \Theta^\kappa F.$$

Hence we obtain the general formula

$$\Delta x^\lambda \Theta^\kappa F = x^\lambda \{(\nu - 2\kappa - \lambda) x + \Theta\} \Theta^\kappa F,$$

by means of which we calculate in succession the values of $\Delta^2 F$, $\Delta^3 F$, ...

Thus,
$$\begin{aligned} \Delta^2 F &= \Delta(\nu x + \Theta) F \\ &= \nu \Delta x F + \Delta\Theta F \\ &= \nu x\{(\nu - 1) x + \Theta\} F + \{(\nu - 2) x + \Theta\}\Theta F \\ &= \{\nu(\nu - 1) x^2 + 2(\nu - 1) x\Theta + \Theta^2\} F. \end{aligned}$$

Hence

$$\begin{aligned} \Delta^3 F &= \nu(\nu - 1)\Delta x^2 F + 2(\nu - 1)\Delta x\Theta F + \Delta\Theta^2 F \\ &= \nu(\nu - 1) x^2\{(\nu - 2) x + \Theta\} F + 2(\nu - 1) x\{(\nu - 3) x + \Theta\}\Theta F \\ &\quad + \{(\nu - 4) x + \Theta\}\Theta^2 F \\ &= \{\nu(\nu - 1)(\nu - 2) x^3 + 3(\nu - 1)(\nu - 2) x^2\Theta + 3(\nu - 2) x\Theta^2 + \Theta^3\} F. \end{aligned}$$

If $[\nu]^n$ is used to denote $\nu(\nu - 1)(\nu - 2)$ .... to n factors ($[\nu]'$ will of course mean ν), we have shown that

$$\begin{aligned} \Delta F &= ([\nu]' x + \Theta) F, \\ \Delta^2 F &= ([\nu]^2 x^2 + 2[\nu - 1]' x\Theta + \Theta^2) F, \\ \Delta^3 F &= ([\nu]^3 x^3 + 3[\nu - 1]^2 x^2\Theta + 3[\nu - 1]' x\Theta^2 + \Theta^3) F, \end{aligned}$$

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*If our sole object were to show that $\Theta F = 0$ is a sufficient as well as necessary condition of the persistence of F, we might dispense with all further calculation. Thus it is obvious that, since $\Delta F = (\nu x + \Theta) F$, $\Delta^n F$ must be of the form $(x, \Theta)^n F$; for the dependence of ν on the degree-weight of the operand will not affect the *form* of Δ^n, but only its numerical coefficients. Hence we conclude that F_1 is of the form $\phi(x, \Theta) F$; and remembering that $\Theta^2 F = 0$, $\Theta^3 F = 0$, ... whenever $\Theta F = 0$, it is at once seen that not only (as was shown in the last lecture) must ΘF vanish when F is persistent under the substitution h, but, conversely, that when $\Theta F = 0$, the altered value of F contains the original value as a factor (the other factor being in this case a function of x only); *i. e.* F is persistent.

and by induction it may be proved that in general

$$\Delta^n F = \left\{ [\nu]^n x^n + n\,[\nu - 1]^{n-1} x^{n-1}\Theta + \frac{n(n-1)}{1.2}[\nu - 2]^{n-2} x^{n-2}\Theta^2 + \ldots . + \Theta^n \right\} F.$$

That the last term of this expression is $\Theta^n F$ is sufficiently obvious; what we wish to prove is that, when m is any positive integer less than n, the term in $\Delta^n F$ which involves Θ^m will be

$$\frac{n(n-1)\ldots .(n-m+1)}{1.2.3\ldots . m}[\nu - m]^{n-m} x^{n-m}\Theta^m F.$$

To find the term involving Θ^m in $\Delta^{n+1}F$, we need only consider the operation of Δ on two consecutive terms of $\Delta^n F$; none of the remaining terms will affect the result. Suppose, then, that

$$\Delta^n F = \ldots . + p x^{n-m}\Theta^m F + q x^{n-m+1}\Theta^{m-1} F + \ldots .$$

Operating with Δ, we find

$$\begin{aligned}\Delta^{n+1} F &= \ldots . + p\Delta x^{n-m}\Theta^m F + q\Delta x^{n-m+1}\Theta^{m-1} F + \ldots . \\ &= \ldots . + p x^{n-m}\{(\nu - n - m)\,x + \Theta\}\,\Theta^m F \\ &\qquad + q x^{n-m+1}\{(\nu - n - m + 1)\,x + \Theta\}\Theta^{m-1} F + \ldots . \\ &= \ldots . + \{p(\nu - n - m) + q\} x^{n+1-m}\Theta^m F + \ldots .\end{aligned}$$

Now, assuming the general term of $\Delta^n F$ to be as written above, we have

$$p = \frac{n(n-1)\ldots .(n-m+1)}{1.2.3\ldots . m}[\nu - m]^{n-m},$$

$$q = \frac{n(n-1)\ldots .(n-m+2)}{1.2.3\ldots .(m-1)}[\nu - m + 1]^{n\;\;m+1};$$

so that
$$q = p\left\{\frac{m(\nu - m + 1)}{n - m + 1}\right\}.$$

Thus the general term of $\Delta^{n+1}F$ has for its numerical coefficient

$$\begin{aligned} p(\nu - n - m) + q &= p\left\{\frac{m(\nu - m + 1) + (\nu - n - m)(n - m + 1)}{n - m + 1}\right\} \\ &= p\left\{\frac{(n+1)(\nu - n)}{n - m + 1}\right\} = \frac{(n+1)n\ldots .(n-m+2)}{1.2.3.m}[\nu - m]^{n+1-m},\end{aligned}$$

which shows that the numerical coefficients in $\Delta^{n+1}F$ obey the same law as those in $\Delta^n F$; and as this law is true for $n = 1, 2, 3$, it is also true universally.

We have thus shown that the general term in

$$\Delta^n F \text{ is } \frac{n(n-1)\ldots .(n-m+1)}{1.2.3\ldots . m}[\nu - m]^{n-m} x^{n-m}\Theta^m F,$$

and, consequently, the corresponding general term in

$$\frac{h^n \Delta^n F}{1.2.3 \ldots . n} \text{ is } \frac{[\nu - m]^{n-m}}{1.2.3 \ldots . (n-m)} h^{n-m} x^{n-m} . \frac{h^m \Theta^m F}{1.2.3 \ldots . m} .$$

Now, as we have already seen,

$$F_1 = \left(1 + h\Delta + \frac{h^2}{1.2}\Delta^2 + \frac{h^3}{1.2.3}\Delta^3 + \ldots .\right) F,$$

which, by merely expressing the symbolic factor as a series of powers of Θ, may be transformed into

$$\begin{aligned} F_1 = & \left(1 + [\nu]'hx + \frac{[\nu]^2}{1.2} h^2x^2 + \frac{[\nu]^3}{1.2.3} h^3x^3 + \ldots .\right) F, \\ & + \left(1 + [\nu - 1]'hx + \frac{[\nu - 1]^2}{1.2} h^2x^2 + \frac{[\nu - 1]^3}{1.2.3} h^3x^3 + \ldots .\right) h\Theta F \\ & + \left(1 + [\nu - 2]'hx + \frac{[\nu - 2]^2}{1.2} h^2x^2 + \frac{[\nu - 2]^3}{1.2.3} h^3x^3 + \ldots .\right) \frac{h^2\Theta^2 F}{1.2} \\ & + \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \end{aligned}$$

where, remembering that $[\nu]^n$ stands for $\nu(\nu - 1)(\nu - 2) \ldots .$ to n factors, it is evident that the functions of x which multiply F, $h\Theta F$, $\frac{h^2}{1.2}\Theta^2 F$, ... are all of them binomial expansions. Hence we immediately obtain

$$\begin{aligned} F_1 & = (1 + hx)^\nu F + (1 + hx)^{\nu-1} h\Theta F + (1 + hx)^{\nu-2} \frac{h^2}{1.2}\Theta^2 F + \ldots . \\ & = (1 + hx)^\nu \left\{1 + (1 + hx)^{-1} h\Theta + (1 + hx)^{-2} \frac{h^2\Theta^2 F}{1.2} + \ldots .\right\} F, \end{aligned}$$

and finally,

$$F_1 = (1 + hx)^\nu e^{\frac{h\Theta}{1 + hx}} F.$$

Mr. Hammond has remarked that, with a slight modification, the foregoing demonstration will serve to establish the analogous theorem, that

$$F_1 = (1 + ht)^{-\mu} e^{-\frac{hV_1}{1 + ht}} F,$$

where, as before, F means any homogeneous and isobaric function of degree i and weight w in the letters $y, t, a, b, c, \ldots .$; and F_1 is what F becomes when, leaving y unaltered, we change x into $x + hy$, where h is any finite quantity. Instead of the operator

$$\Theta = -y\partial_t + a\partial_b + 2b\partial_c + 3c\partial_d + \ldots . = -y\partial_t + \Omega$$

we have $\quad -V_1 = yt\partial_y + t^2\partial_t - 2a^2\partial_b - 5ab\partial_c - \ldots . = yt\partial_y + t^2\partial_t - V;$*

* This theorem was stated without proof in Lecture VIII, where, through inadvertence, the term $yt\partial_y$ in the expression for V_1 was omitted.

and instead of $\nu = 3i + 2w$, a different number, $\mu = 3i + w$ (which I have called the characteristic), taken negatively.

If we suppose that

F_1 is what F becomes on changing x into $x + hy$,

and F_2 " " F " " " x " $x + \varepsilon y$,

then F_2 " " F " " " x " $x + (h + \varepsilon) y$.

Hence, if $$F_1 = F + hP + \frac{h^2}{1.2} Q + \frac{h^3}{1.2.3} R + \ldots,$$

we must have $$F_2 = F + (h + \varepsilon) P + \frac{(h+\varepsilon)^2}{1.2} Q + \frac{(h+\varepsilon)^3}{1.2.3} R + \ldots$$
$$= F_1 + \varepsilon \frac{dF_1}{dh} + \ldots$$

Thus, if ε be regarded as infinitesimal, and we write

$$\frac{F_2 - F_1}{\varepsilon} = \Delta F_1,$$

it follows that $$\Delta F_1 = P + hQ + \frac{h^2}{1.2} R + \ldots$$

But, by the direct operation of Δ, we find

$$\Delta F_1 = \Delta F + h\Delta P + \frac{h^2}{1.2} \Delta Q + \ldots,$$

and, comparing these two values of ΔF_1,

$$P = \Delta F,$$
$$Q = \Delta P = \Delta^2 F,$$
$$R = \Delta Q = \Delta^3 F,$$
$$\ldots\ldots\ldots\ldots$$

Hence it follows that

$$F_1 = F + h\Delta F + \frac{h^2}{1.2} \Delta^2 F + \frac{h^3}{1.2.3} \Delta^3 F + \ldots$$

It remains to find the value of $\Delta^n F$. This can be effected by means of formulae given in Lecture VIII (*American Journal of Mathematics*, Vol. VIII, p. 245), where it is shown that

$$\Delta x = y,$$
$$\Delta y = 0,$$
$$\Delta t = -t^2,$$
$$\Delta a = -3at,$$
$$\Delta b = -4bt - 2a^2,$$
$$\Delta c = -5ct - 5ab,$$
$$\Delta d = -6dt - 6ac - 3b^2,$$
$$\Delta e = -7et - 7ad - 7bc,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

We now show that

$$\Delta F = -(\mu t + V_1) F,$$

where
$$V_1 = V - t^2\partial_t - yt\partial_y,$$

just as in the cognate theorem we had

$$\Delta F = (\nu x + \Theta) F.$$

Since F is a function of $y, t, a, b, c, \ldots.$ without x, it is evident that

$$\begin{aligned}\Delta F &= \frac{dF}{dy}\Delta y + \frac{dF}{dt}\Delta t + \ldots. \\ &= -t(t\partial_t + 3a\partial_a + 4b\partial_b + 5c\partial_c + \ldots.) F \\ &\quad - \{2a^2\partial_b + 5ab\partial_c + (6ac + 3b^2)\partial_d + \ldots.\} F,\end{aligned}$$

where the part of ΔF which is independent of t is $-VF$.

Now,
$$y\partial_y + t\partial_t + a\partial_a + b\partial_b + c\partial_c + \ldots. = i$$
and
$$-2y\partial_y - t\partial_t + b\partial_b + 2c\partial_c + \ldots. = w;$$
so that
$$t\partial_t + 3a\partial_a + 4b\partial_b + 5c\partial_c + \ldots. = 3i + w - y\partial_y - t\partial_t.$$

Hence, writing $3i + w = \mu$,

$$\begin{aligned}\Delta F &= -t(\mu - y\partial_y - t\partial_t) F - VF \\ &= -(\mu t + V_1) F,\end{aligned}$$

where
$$V_1 = V - t^2\partial_t - yt\partial_y.$$

Observing that $V_1^\kappa F$ is of degree $i + \kappa$ and weight $w - \kappa$; since

$$3(i + \kappa) + (w - \kappa) = \mu + 2\kappa,$$

we see that
$$\Delta V_1^\kappa F = -\{(\mu + 2\kappa) t + V_1\} V_1^\kappa F.$$

Again,
$$\begin{aligned}\Delta t^\lambda V_1^\kappa F &= \lambda t^{\lambda-1} V_1^\kappa F . \Delta t + t^\lambda \Delta V_1^\kappa F \\ &= -\lambda t^{\lambda+1} V_1^\kappa F - t^\lambda \{(\mu + 2\kappa) t + V_1\} V_1^\kappa F.\end{aligned}$$

We thus obtain the formula

$$\Delta t^\lambda V_1^\kappa F = -t^\lambda \{(\mu + \lambda + 2\kappa) t + V_1\} V_1^\kappa F, \tag{1}$$

analogous to the one previously employed,

$$\Delta x^\lambda \Theta^\kappa F = x^\lambda \{(\nu - 2\kappa - \lambda) x + \Theta\} \Theta^\kappa F. \tag{2}$$

The remainder of the work will be step for step the same for this as for the previous theorem. In fact, by using (1) just as we used (2), we shall deduce

$$F_1 = (1 + ht)^{-\mu} e^{-\frac{hV_1}{1+ht}} F, \tag{3}$$

just as we deduced the analogous formula

$$F_1 = (1 + hx)^\nu e^{\frac{h\Theta}{1+hx}} F. \tag{4}$$

The reason of this is obvious: by interchanging x and t, μ and $-\nu$, Θ and $-V_1$, we interchange the formulae (1) and (2), (3) and (4).

It may be well to observe that if we use S_h to denote a substitution of such a nature that
$$S_\epsilon S_h = S_{h+\epsilon},$$
and if (regarding ε as an infinitesimal) we write
$$\frac{S_\epsilon - 1}{\varepsilon} = \Delta,$$
then in general
$$S_h F = e^{h\Delta} F.$$
The proof of this proposition is virtually contained in what precedes.

LECTURE XXIV.

Whenever a rational integral function of $x, y, t, a, b, c, \ldots$ is persistent in form under the general linear substitution, it cannot contain explicitly either x, y or t, but must be a function of the remaining letters $a, b, c, \ldots$ (the successive modified derivatives, beginning with the second, of y with respect to x) alone.

For if, keeping y unaltered, we change x into $x + \alpha$, where α is any arbitrary constant which may be regarded as an infinitesimal, the derivatives $t, a, b, c, \ldots$ are not affected by this change, and consequently the function
$$F = F(x, y, t, a, b, c, \ldots) \text{ becomes } F + \alpha\frac{dF}{dx},$$
which cannot be divisible by F unless $\dfrac{dF}{dx} = 0$.

(The alternative hypothesis of $\dfrac{dF}{dx}$ being divisible by F is inadmissible, because F is a rational integral function.)

Hence F cannot contain x explicitly; and if we write $y + \beta$ for y, keeping x unchanged, we see, in like manner, that F cannot contain y explicitly.

Again, if in the function
$$F = F(t, a, b, c, \ldots)$$
we change x, y into $x + \alpha$, $y + \beta_{\prime} x + \beta$, the effect of this substitution will be to increase t by the arbitrary constant $\beta_{\prime}$, without altering any of the remaining derivatives $a, b, c, \ldots$

Hence, in order that the form of F may still be persistent, we must have $\frac{dF}{dt}=0$; the reasoning being just the same as that by which $\frac{dF}{dx}$ was seen to vanish. Thus, F does not contain t explicitly. Moreover, the function

$$F = F(a, b, c, \ldots .)$$

must be both homogeneous and isobaric.

For the substitution of $\alpha_{\prime}x + \alpha$, $\beta_{\prime\prime}y + \beta_{\prime}x + \beta$ for x, y, respectively, will multiply the letters

$$\begin{array}{lllll} a\ , & b\ , & c\ , & d\ , & \ldots . \end{array}$$

by

$$\begin{array}{lllll} \beta_{\prime\prime}\alpha_{\prime}^{-2}, & \beta_{\prime\prime}\alpha_{\prime}^{-3}, & \beta_{\prime\prime}\alpha_{\prime}^{-4}, & \beta_{\prime\prime}\alpha_{\prime}^{-5}, & \ldots . \end{array}$$

Each term of F will therefore be multiplied by a positive power of $\beta_{\prime\prime}$ and a negative power of $\alpha_{\prime}$.

Let one of the terms of F be $a^{\lambda_0}b^{\lambda_1}c^{\lambda_2}d^{\lambda_3} \ldots .$ It will be multiplied by

$$\beta_{\prime\prime}{}^{\lambda_0+\lambda_1+\lambda_2+\lambda_3+\ldots .}\alpha_{\prime}{}^{-(2\lambda_0+3\lambda_1+4\lambda_2+5\lambda_3+\ldots .)}.$$

In order that F may retain its form, this multiplier must be the same for every term of F, no matter what arbitrary values are assigned to $\alpha_{\prime}$ and $\beta_{\prime\prime}$. This can only happen when, for all terms of the function F, we have

$$\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \ldots . = \text{const.}$$

and

$$\lambda_1 + 2\lambda_2 + 3\lambda_3 + \ldots . = \text{const.},$$

i. e. when F is homogeneous and isobaric.

We have thus proved that among all the rational integral functions of $x, y, t, a, b, c, \ldots .$ the only ones persistent under the substitution of $\alpha + \alpha_{\prime}x$, $\beta + \beta_{\prime}x + \beta_{\prime\prime}y$ for x, y, respectively, are such as simultaneously satisfy the conditions of not explicitly containing x, y or t, and of being homogeneous and isobaric in the remaining letters $a, b, c, \ldots .$

If F, any function satisfying these conditions, merely acquires an extraneous factor when, leaving y unaltered, we change x into $x + hy$, the form of F will be persistent under the general linear substitution. For both $\alpha + \alpha_{\prime}(x + hy)$ and $\beta + \beta_{\prime}(x + hy) + \beta_{\prime\prime}y$ are general linear functions of x, y, 1.

Now, the change of x into $x + hy$ converts (as was shown in the preceding lecture) F into

$$F_1 = (1 + ht)^{-\mu}e^{-\frac{hV_1}{1+ht}}F,$$

where

$$V_1 = V - t^2\partial_t - yt\partial_y.$$

But, since neither y nor t occurs in F, we must have

$$\partial_y F = 0 \text{ and } \partial_t F = 0.$$

Consequently,

$$V_1 F = VF,\ V_1^2 F = V^2 F,$$

and so on. Hence

$$F_1 = (1+ht)^{-\mu} e^{-\frac{hV}{1+ht}} F$$
$$= (1+ht)^{-\mu} F - (1+ht)^{-\mu-1} h VF + (1+ht)^{-\mu-2} \frac{h^2 V^2}{1.2} F - \ldots.$$

Unless VF, V^2F, V^3F, ... all of them vanish, F_1 cannot contain F as a factor. If it could, VF, V^2F, ... would all have to be divisible by F. But this is impossible; for VF, a rational integral function of a, b, c, ... whose weight is $w-1$, cannot be divisible by F, a rational integral function of weight w.

We must therefore have

$$VF = 0 \qquad \text{(which implies } V^2F = 0\text{, etc.)}$$

as the necessary and sufficient condition of the persistence of the form of F under the general linear substitution. In other words, F must be a pure reciprocant.

In order that F may also be persistent in form under the general homographic substitution, it must (besides being a pure reciprocant) be subject to annihilation by the operator

$$\Omega = a\partial_b + 2b\partial_c + 3c\partial_d + \ldots.$$

For it was seen, in the preceding lecture, that the special homographic substitution in which $\frac{x}{1+hx}$, $\frac{y}{1+hx}$ are written instead of x, y, respectively, has the effect of changing any homogeneous and isobaric function F into F_1, where

$$F_1 = (1+hx)^{\nu} e^{\frac{h\Theta}{1+hx}} F,$$
$$\Theta = \Omega - y\partial_t.$$

When the letter t does not occur in F, we may write $\partial_t F = 0$, so that Θ becomes simply Ω, and the above formula becomes

$$F_1 = (1+hx)^{\nu} e^{\frac{h\Omega}{1+hx}} F.$$

Hence it follows immediately that, when F is a rational integral function of the letters a, b, c, ..., the condition $\Omega F = 0$ is sufficient as well as necessary to ensure the persistence of the form of F under the special homographic substitution we have employed.

But when F is a pure reciprocant it also satisfies the condition $VF = 0$, and it is the simultaneous satisfaction of $\Omega F = 0$ and $VF = 0$ that ensures the persistence of the form of F under the most general homographic substitution.

This may be shown by combining the substitution $\frac{x}{1+hx}$, $\frac{y}{1+hx}$ (for which F is persistent when, and only when, $\Omega F=0$) with the general linear substitution (for which $VF=0$ is the necessary and sufficient condition of the persistence of the form of F), so as to obtain the most general homographic substitution. Thus the linear substitution

$$\left.\begin{aligned} x &= lx_{,} + my_{,} + n \\ y &= l'x_{,} + m'y_{,} + n' \end{aligned}\right\},$$

when combined with

$$x_{,} = \frac{x_{,,}}{1+hx_{,,}}, \quad y_{,} = \frac{y_{,,}}{1+hx_{,,}},$$

gives the substitution

$$\left.\begin{aligned} x &= \frac{lx_{,,}+my_{,,}+n(1+hx_{,,})}{1+hx_{,,}} \\ y &= \frac{l'x_{,,}+m'y_{,,}+n'(1+hx_{,,})}{1+hx_{,,}} \end{aligned}\right\},$$

in which both the numerators are general linear functions.

By combining the substitution just obtained with the linear substitution

$$x_{,,}=\lambda x_{,,,}+\mu y_{,,,}+\nu, \quad y_{,,}=y_{,,,},$$

the denominator of each fraction is changed into a general linear function, and thus, by combining the special homographic substitution $\frac{x}{1+hx}$, $\frac{y}{1+hx}$ with two linear substitutions, we arrive at the most general homographic substitution.

This proves that the necessary and sufficient condition of F being a *homographically persistent* form is the coexistence of the two conditions

$$VF=0, \quad \Omega F=0.$$

Thus a Projective Reciprocant, or Principiant, or Differential Invariant, combines the natures of a Pure Reciprocant and Invariant *in respect of the elements*.

Notice that every Pure Reciprocant is an Invariant of the Reciprocal Function $\left(i.\ e.\right.$ the numerator of the expression for $\frac{d^nx}{dy^n}$ in terms of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, ... , or what is the same in terms of the modified derivatives $\left.t, a, b, \ldots\right)$, but the elements of such invariants are *not* the original simple elements, but more or less complicated functions of them.

What has just been stated is obvious from the fact that all invariants of the "reciprocal function" have been shown to be pure reciprocants (*vide* Lect. XIX).

The ordinary protomorph invariants of this function will have for their leading term a power of a multiplied by a single letter. Consequently, by reasoning previously employed in these lectures, every pure reciprocant will be a rational function of invariants of the Reciprocal Function divided by some power of a. Thus, for example, the Reciprocal Function

$$14a^4 - 21a^2bt + 3(2ac + b^2)t^2 - dt^3 = (\alpha, \beta, \gamma, \delta \between 1, -t)^3$$

if

$$\alpha = 14a^4, \ \beta = 7a^2b, \ \gamma = 2ac + b^2, \ \delta = d.$$

The two protomorph invariants of this reciprocal function are

$$\alpha\gamma - \beta^2 = 7a^4(4ac - 5b^2)$$

and

$$\alpha^2\delta - 3\alpha\beta\gamma + 2\beta^3 = 196a^6(a^2d - 3abc + 2b^3).$$

All other pure reciprocants of extent 3 may be rationally expressed in terms of a and the two protomorphs $4ac - 5b^2$, $a^2d - 3abc + 2b^3$; *i. e.* all pure reciprocants of extent 3 are invariants of the reciprocal function of extent 3.

The reasoning employed can be applied with equal facility to the general case of extent n.

Instead of $\frac{x}{1+hx}$, $\frac{y}{1+hx}$, let us consider the special homographic substitution $\frac{1}{x}$, $\frac{y}{x}$ employed by M. Halphen.

Writing

$$X = \frac{1}{x} \text{ and } Y = \frac{y}{x},$$

let $Y_1, Y_2, Y_3, \ldots$ denote the successive derivatives of Y with respect to X, and $y_1, y_2, y_3, \ldots$ those of y with respect to x. Then

$$\begin{aligned}
Y &= x^{-1}y, \\
Y_1 &= -x\left(y_1 - \frac{1}{x}y\right), \\
Y_2 &= x^3y_2, \\
Y_3 &= -x^5\left(y_3 + \frac{3}{x}y_2\right), \\
Y_4 &= x^7\left(y_4 + \frac{8}{x}y_3 + \frac{12}{x^2}y_2\right), \\
Y_5 &= -x^9\left(y_5 + \frac{15}{x}y_4 + \frac{60}{x^2}y_3 + \frac{60}{x^3}y_2\right), \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Hence, if $a, b, c, d, \ldots$ are the successive modified derivatives (beginning with

the second) of y with respect to x, and $a', b', c', d', \ldots.$ the corresponding modified derivatives of Y with respect to X, it follows immediately that

$$\begin{aligned}
a' &= \quad x^3 a, \\
b' &= -x^5\left(b + \frac{1}{x}a\right), \\
c' &= \quad x^7\left(c + \frac{2}{x}b + \frac{1}{x^2}a\right), \\
d' &= -x^9\left(d + \frac{3}{x}c + \frac{3}{x^2}b + \frac{1}{x^3}a\right), \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Attributing the weights $0, 1, 2, 3, \ldots.$ to the letters $a, b, c, d, \ldots.$, it is very easily seen that if F is any homogeneous and isobaric function of degree i and weight w,

$$F(a', b', c', \ldots.) = (-)^w x^{3i+2w} F\left(a, b + \frac{1}{x}a, c + \frac{2}{x}b + \frac{1}{x^2}a, \ldots.\right).$$

But we proved (in Lecture XXII) that for all values of h

$$F(a, b + ah, c + 2bh + ah^2, \ldots.) = e^{h\Omega} F(a, b, c, \ldots.).$$

Hence, making $h = \frac{1}{x}$, we obtain

$$F(a', b', c', d', \ldots.) = (-)^w x^{3i+2w} e^{\frac{\Omega}{x}} F(a, b, c, \ldots.),$$

which proves that the satisfaction of

$$\Omega F(a, b, c, \ldots.) = 0$$

is the necessary and sufficient condition for the persistence of the form of F under the Halphenian substitution $\frac{1}{x}, \frac{y}{x}$.

Similarly we might prove that $F(y, t, a, b, c, \ldots.)$, which contains y and t, but not x, is changed by the substitution $\frac{1}{x}, \frac{y}{x}$ into

$$(-)^w x^\nu e^{\frac{\Theta}{x}} F(y, t, a, b, c, \ldots.),$$

where $$\Theta = -y\partial_t + a\partial_b + 2b\partial_c + \ldots. = \Omega - y\partial_t;$$

or we may deduce this result from the formula, demonstrated in the preceding lecture of this course,

$$F_1 = (1 + hx)^\nu e^{\frac{h\Theta}{1+hx}} F,$$

in which F_1 is what F becomes in consequence of the substitution $\frac{x}{1+hx}, \frac{y}{1+hx}$ impressed on the variables.

Let i be the degree and ω the weight measured by the sum of the orders of differentiation in each term of

$$F(y, t, a, b, c, \ldots).$$

If we measure the weight by the sum of the orders of differentiation of every term of F diminished by 2 units for each letter in the term, then

$$w = \omega - 2i \text{ and } 2\omega - i = 3i + 2w = \nu.$$

Let $\quad F(y, t, a, b, c, \ldots)$ become $F'(y, t, a, b, c, \ldots)$,
when we change

$$x \text{ into } qx + p \text{ and } y \text{ into } ry;$$

then $\quad F'(y, t, a, b, c, \ldots) = r^i q^{-\omega} F(y, t, a, b, c, \ldots).$

A further substitution $\dfrac{x}{1+hx}$, $\dfrac{y}{1+hx}$, impressed on the variables in F', will convert the original variables into

$$\frac{qx}{1+hx} + p \text{ and } \frac{ry}{1+hx},$$

i. e. into $$\frac{p(1+hx)+qx}{1+hx} \text{ and } \frac{ry}{1+hx}.$$

The function F' is at the same time changed into

$$r^i q^{-\omega}(1+hx)^\nu e^{\frac{h\Theta}{1+hx}} F(y, t, a, b, c, \ldots).$$

If now, in the above, we write $p = h$, $q = -h^2$, $r = h$, we shall have changed the original variables x, y into $\dfrac{h}{1+hx}$, $\dfrac{hy}{1+hx}$, and the original function F into

$$h^i(-h^2)^{-\omega}(1+hx)^\nu e^{\frac{h\Theta}{1+hx}} F = (-)^\omega h^{i-2\omega}(1+hx)^\nu e^{\frac{h\Theta}{1+hx}} F = (-)^w \left(\frac{1+hx}{h}\right)^\nu e^{\frac{h\Theta}{1+hx}} F.$$

Let h become infinite; then $\dfrac{h}{1+hx}$, $\dfrac{hy}{1+hx}$ and $(-)^w\left(\dfrac{1+hx}{h}\right)^\nu e^{\frac{h\Theta}{1+hx}} F$ become $\dfrac{1}{x}$, $\dfrac{y}{x}$ and $(-)^w x^\nu e^{\frac{\Theta}{x}} F$, showing that the substitution $\dfrac{1}{x}$, $\dfrac{y}{x}$ changes F into $(-)^w x^\nu e^{\frac{\Theta}{x}} F$.

(*To be continued.*)